Journal of Statistical Physics, Vol. 108, Nos. 3/4, August 2002 (© 2002)

Multi-Fractal Formalism for Quasi-Self-Similar Functions

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Received January 12, 2001; accepted December 12, 2001

The study of multi-fractal functions has proved important in several domains of physics. Some physical phenomena such as fully developed turbulence or diffusion limited aggregates seem to exhibit some sort of self-similarity. The validity of the multi-fractal formalism has been proved to be valid for self-similar functions. But, multi-fractals encountered in physics or image processing are not exactly self-similar. For this reason, we extend the validity of the multi-fractal formalism for a class of some non-self-similar functions. Our functions are written as the superposition of "similar" structures at different scales, reminiscent of some possible modelization of turbulence or cascade models. Their expressions look also like wavelet decompositions. For the computation of their spectrum of singularities, it is unknown how to construct Gibbs measures. However, it suffices to use measures constructed according the Frostman's method. Besides, we compute the box dimension of the graphs.

KEY WORDS: Multi-fractal formalism; wavelets; turbulence; cascade models; Gibbs measures; non-self-similar functions; Frostman's method; box dimension.

1. INTRODUCTION

A bounded function $F: \mathbb{R}^m \to \mathbb{C}$ is $C^\alpha(x_0)$ for $\alpha > 0$ if there exists a polynomial P of degree at most $[\alpha]$ and a constant C such that, if $|x-x_0| \leqslant 1$,

$$|F(x)-P(x-x_0)| \leqslant C\,|x-x_0|^\alpha \tag{1}$$

A function F belongs to $C^\alpha(\mathbb{R}^m)$ if (1) holds for any x and x_0 in $\mathbb{R}^m$ with a uniform constant C.

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0022-4715/02/0800-0541/0

In order to characterize both the regularity and the irregularity of F at x_0, we define the Hölder exponent $\alpha_F(x_0)$ of F at x_0 as the supremum of all values of α such that F is $C^\alpha(x_0)$. If F is n times continuously differentiable at the point x_0 then one can use for the polynomial $P(x-x_0)$ the order n Taylor series of F at x_0 and thus prove that $\alpha_F(x_0) \geqslant n$. Thus the Hölder exponent $\alpha_F(x_0)$ measures how irregular F is at the point x_0. The higher the exponent $\alpha_F(x_0)$, the more regular the function F.

A function F is multi-fractal if $\alpha_F(x)$ differs widely from point to point. In this case, the determination of the Hölder exponents $\alpha_F(x)$ is difficult. Nonetheless the study of such functions has proved important in several domains of physics and signal analysis (for example, see, ref. 1).

The determination of the Hölder exponent of a function can be reduced to estimating its wavelet transform near x_0, using either Proposition 1 or the discrete form of Proposition 1 (see refs. 2–4). Let ψ be a wavelet, i.e., a $C^k(\mathbb{R}^m)$ function where all moments of order less than k vanish and all derivatives of order less than k are well localized (and k is large enough depending on the properties of F we want to analyze). The wavelet transform of F at the position $b \in \mathbb{R}^m$ and for the scale $a > 0$ is

$$C_{a,b}(F) = \frac{1}{a^m} \int_{\mathbb{R}^m} F(t)\, \bar{\psi}\left(\frac{t-b}{a}\right) dt \tag{2}$$

Proposition 1. Let $\alpha < k$.

- $F \in C^\alpha(\mathbb{R}^m)$ if and only if $|C_{a,b}(F)| \leqslant C a^\alpha$ for all b and sufficiently small a.
- If $F \in C^\alpha(x_0)$, then for sufficiently small a and $|b-x_0| \leqslant 1/2$,

$$|C_{a,b}(F)| \leqslant C a^\alpha \left(1 + \frac{|b-x_0|}{a}\right)^\alpha \tag{3}$$

- If (3) holds and if $F \in C^\varepsilon(\mathbb{R}^m)$ for an $\varepsilon > 0$, then there exists a polynomial P such that, if $|x-x_0| \leqslant 1/2$,

$$|F(x) - P(x-x_0)| \leqslant C\, |x-x_0|^\alpha \log\left(\frac{2}{|x-x_0|}\right) \tag{4}$$

The pointwise Hölder regularity is summed up by computing the spectrum of singularities $d(\alpha)$ which associates to each α the Hausdorff dimension $d(\alpha)$ of the set E_F^α of points x where $\alpha_F(x) = \alpha$ (conventionally the dimension of the empty set is $-\infty$). A function is called multi-fractal when $d(\alpha)$ is defined at least on an interval of non-empty interior.

Multi-fractal analysis started to be developed in the context of fully developed turbulence. Mandelbrot first introduced cascade models for the dissipation of energy in a turbulent fluid (see refs. 5–7), that turned out to be multi-fractal measures, see ref. 8 and references therein. This remarkable insight did meet the experimental results obtained in wind-tunnels which show that the regularity of the velocity of a turbulent fluid fluctuates widely from point to point (see ref. 9). This phenomenon, related to intermittency, suggests that the spectrum of singularities of the velocity of the fluid might be a universal function, in which case its determination would yield a fundamental information on the nature of turbulence.

Our purpose is to check the validity of an heuristic argument, that relates the spectrum of singularities of a function, based on the wavelet transform. The Frish–Parisi conjecture (due also to Arneodo *et al.*) which is called the multi-fractal formalism for functions (see refs. 10 and 11) asserts that

$$d(\alpha)=\inf_{p}\;(\alpha p-\eta(p)+m) \tag{5}$$

In (5), the function $\eta(p)$ is the Besov exponent given by

$$\eta(p)=\sup\{s\colon F\in B_p^{s/p,\infty}(\mathbb{R}^m)\} \tag{6}$$

where the Besov space $B_p^{s,\infty}(\mathbb{R}^m)$ (cf. refs. 12 and 13) is the set of functions F such that

$$\int_{\mathbb{R}^m}|C_{a,b}(F)|^p\,db\leqslant Ca^{sp}\qquad\text{for all } a \text{ small enough} \tag{7}$$

Remark 1. It follows from above that to compute both the Hölder exponent $\alpha_F(x)$ for any point x and the Besov exponent $\eta(p)$ we should estimate the size of the wavelet transform $C_{a,b}(F)$ everywhere, and we should obtain estimations of the form (3) and estimations of the form (7). In general, that estimations are very difficult to obtain. However, if F satisfies some self-affinity conditions then its wavelet transform satisfies a similar self-affinity, for example if $F\colon \mathbb{R}^m\mapsto\mathbb{R}$ satisfies

$$\exists r>0,\;\exists\lambda/\forall x\qquad F(x)=\lambda F(rx) \tag{8}$$

then its wavelet transform satisfies

$$\forall a>0,\;\forall b\qquad C_{a,b}(F)=\lambda C_{ra,rb}(F) \tag{9}$$

By the same way, if F is T-periodic, i.e.,

$$\forall x \qquad F(x+T)=F(x) \tag{10}$$

then

$$\forall a>0,\ \forall b \qquad C_{a,b}(F)=C_{a,b+T}(F) \tag{11}$$

Relations (9) and (11) allow the estimation of the size of the wavelet transform $C_{a,b}(F)$ everywhere. Besides, from an experimental point of view, the wavelet transform $C_{a,b}(F)$ of a signal or an image F can be estimated on a finite range of scales, so if the signal (or the image) satisfies the self-affinity conditions (8) and (10) then its wavelet transform $C_{a,b}(F)$ can be estimated at all scales.

Each time the multi-fractal formalism has been shown to hold, it was the consequence of some self-similarity (deterministic or statistic) either for the function or of its wavelet transform. It is therefore reasonable to conjecture that if a function satisfies some self-similarity condition, then the multi-fractal formalism is likely to hold. But it is impossible to state a reasonably general conjecture (one should be careful to avoid in such a statement the counterexamples exhibited by Ben Slimane in refs. 14–16).

Some multi-fractals are partly self-similar. The applications of iterated functional systems have shown the importance of such fractals in images processing.[17] Some physical phenomena such as fully developed turbulence[18] or diffusion limited aggregates[19] also seem to exhibit some sort of self-similarity.

The validity of the Frish–Parisi conjecture has been first proved for a restrictive class of self-similar (or self-affine) functions (see refs. 10, 20, and 21). A self-similar (or self-affine) function (in the sense of Jaffard[12]) is a function which, modulo an error function g which is more regular, is invariant under specific transformations involving mainly linear contractions (as in (8)) and translations (as in (10)). This means that F satisfies a functional equation of the form

$$F(x)=\sum_{j=1}^{d} \lambda_j F(S_j^{-1}(x))+g(x) \tag{12}$$

where the S_j are contractions on a bounded open set Ω, and the $|\lambda_j|$ are smaller than 1.

In ref. 12, Jaffard proved the multi-fractal formalism for self-similar functions in the case where

- the S_i are contractive similitudes (i.e., the product of an isometry with a homothety of ratio < 1) such that

$$S_i(\Omega) \subset \Omega \tag{13}$$

and

$$S_i(\Omega) \cap S_j(\Omega) = \varnothing \qquad \text{if} \quad i \neq j \quad \text{(separation condition)} \tag{14}$$

- The function g is C^k with all derivatives of order less than k having fast decay.
- There exists $x_0 \in \Omega$ such that $F \notin C^k(x_0)$.

In ref. 22, Ben Slimane extended the results of Jaffard to nonlinear self-similar functions associated to hyperbolic dynamical systems; he replaced similitudes S_j by nonlinear contractions T_j in dimension $m = 1$, and T_j's that are analytic mappings of $z = x + \mathbf{i}y$ ($\mathbf{i}$ is such that $\mathbf{i}^2 = -1$) in dimension $m = 2$. The fundamental idea behind these extensions is to make some extra hypothesis which imply that locally these contractions are close to linear contractions in dimension one and "contract with the same rate" in each direction in dimension two.

The author of ref. 15 also proved that the multi-fractal formalism no longer holds in dimension $m \geqslant 2$ if the S_i's contract at different rates in each direction of $\mathbb{R}^m$, then he showed how to modify this formalism to suit this type of functions.

Let us recall briefly the results of ref. 22; contrary to similitudes, the nonlinear contractions T_j are only defined on a bounded open set Ω of $\mathbb{R}^m$ (where $m = 1$ or $m = 2$), so the assumption that g has fast decay at infinity does not make any sense, and has been replaced by the fact that g must be supported in $\bar{\Omega}$. The sense of the self-similarity (12) for the T_j's becomes

$$F(x) = \begin{cases} \lambda_j F(T_j^{-1}(x)) + g(x) & \text{if} \quad x \in T_j(\Omega) \\ g(x) & \text{if} \quad x \notin \bigcup_{j=1,\ldots,d} T_j(\Omega) \end{cases} \tag{15}$$

For the sake of simplicity, we only treat the case $m = 1$, $\Omega =]0, 1[$ and $d = 2$. The general case (i.e., $m = 1$ or 2, Ω is a bounded open set of $\mathbb{R}^m$ and $d \geqslant 2$) is similar. Suppose that the T_j are C^{k+1} on $I = \bar{\Omega} = [0, 1]$ and satisfy:

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$$T_j(I) \subset \Omega \qquad \forall j = 1, 2 \tag{16}$$

$$T_1(I) \cap T_2(I) = \varnothing \tag{17}$$

- there exists constants θ and ρ such that

$$0 < \theta \leqslant |T'_j(x)| \leqslant \rho < 1 \qquad \forall j = 1, 2 \quad \text{and} \quad \forall x \in I \tag{18}$$

Assume that g is a C^k function supported in I and that there exists $x_0 \in]0, 1[$ such that $F \notin C^k(x_0)$.

The fundamental idea behind the extensions of ref. 22 is that, the extra hypothesis (18) and the fact that T''_1 and T''_2 are bounded imply that locally the nonlinear contractions $T_{i_1} \circ \cdots \circ T_{i_n}$ can be uniformly approximated by similitudes in the following sense (see refs. 23–25).

For $i = (i_1,..., i_n) \in \{1, 2\}^n$, denote $|i| = n$, $T_i = T_{i_1} \circ \cdots \circ T_{i_n}$, $I_i = T_i(I)$ and $|I_i| = \text{diam}(I_i)$.

Lemma 1. There exists a constant $\mathscr{D} \geqslant 1$ such that

$$\mathscr{D}^{-1}\, |I_i| \leqslant |T'_i(x)| \leqslant \mathscr{D}\, |I_i| \tag{19}$$

$\forall x \in I$, $\forall i = (i_1,..., i_n) \in \{1, 2\}^n$ and $n \in \mathbb{N}^*$.

As a consequence of Lemma 1 and the mean value theorem, we have the following lemma (cf. refs. 23–25) (often called distortion lemma).

Lemma 2. There exists a positive number $\mathscr{D}$ such that for any branches $i = (i_1,..., i_n)$ and $j = (j_1,..., j_{n'})$

$$\mathscr{D}^{-1}\, |I_i|\, |I_j| \leqslant |I_{ij}| \leqslant \mathscr{D}\, |I_i|\, |I_j|$$

Iterating (12) or (15), F satisfies for any integer $N \in \mathbb{N}^*$ and $x \in \mathbb{R}$

$$F(x) = \sum_{n=0}^{N-1} \sum_{i = (i_1,..., i_n) \in \{1, 2\}^n} \lambda_{i_1} \cdots \lambda_{i_n}\, g(T_i^{-1}(x)) + \sum_{i = (i_1,..., i_N) \in \{1, 2\}^N} \lambda_{i_1} \cdots \lambda_{i_N}\, F(T_i^{-1}(x)) \tag{20}$$

(here by convention $g(T_i^{-1}(x)) = 0$ for $x \notin I_i$ and the term of the series associated to $n = 0$ is $g(x)$).

The nonlinear self-similar equation (15) has a unique solution which is a bounded function supported in I. It is given by the series

$$F(x) = \sum_{n=0}^{\infty} \sum_{i = (i_1,..., i_n) \in \{1, 2\}^n} \lambda_{i_1} \cdots \lambda_{i_n}\, g(T_i^{-1}(x)) \tag{21}$$

By adding a simple condition on the λ_j, the function F has a positive global Hölder regularity (i.e., $F \in C^{\alpha_{\min}}(\mathbb{R})$ with $\alpha_{\min} > 0$).

Denote by K the unique non-empty compact set satisfying $K = T_1(K) \cup T_2(K)$ (see ref. 26). It follows from the separation hypothesis (17) that there is a natural bijection π from the tree $\{1, 2\}^{\mathbb{N}}$ to K, given by

$$\{\pi(i_1,..., i_n,...)\} = \{\lim_{n \mapsto \infty} T_{i_1} \circ \cdots \circ T_{i_n}(t)\} \tag{22}$$

The value of $\pi(i_1,..., i_n,...)$ is independent of the initial value t. The sequence $(i_1,..., i_n,...)$ is called the code of $\pi(i_1,..., i_n,...)$.

In refs. 12 and 22, for $x \notin K$ the function F is C^k on a neighbourhood of x, however for $x \in K$

$$\text{if} \quad a(x) := \liminf_{n \to \infty} \frac{\log |\lambda_{i_1(x)} \cdots \lambda_{i_n(x)}|}{\log |I_{i_1(x) \cdots i_n(x)}|} < k \qquad \text{then} \quad \alpha_F(x) = a(x)$$

For the computation of the spectrum of singularities $d(\alpha)$, the authors of refs. 12 and 22 used Gibbs measures. Let μ_n be the probability measure on $[0, 1]$ which associates the weight $\frac{|\lambda_{i_1} \cdots \lambda_{i_n}|}{(|\lambda_1| + |\lambda_2|)^n}$ for each $I_{(i_1,..., i_n)}$. Clearly, we obtain at the limit $n \longrightarrow \infty$ a probability measure μ supported on K. For any $n \geqslant 1$, $\mathscr{T}_n := \{I_i;\ |i| = n\}$ is a partition of K. Let $\mathscr{T} = \bigcup_{n \geqslant 1} \mathscr{T}_n$. For $E \subset K$ and $s > 0$, denote

$$H^s(E) = \lim_{\varepsilon \to 0} \left(\inf \left\{ \sum |I_i|^s;\ E \subset \bigcup_i I_i \text{ and } |I_i| \leqslant \varepsilon \right\} \right)$$

and

$$\dim(E) = \inf\{s > 0;\ H^s(E) = 0\}$$

So the index dim is defined in a similar way to Hausdorff dimension, but by considering only coverings by elements of $\mathscr{T}$.

Thanks to the fact that for any branches $i = (i_1,..., i_l)$ and $j = (j_1,..., j_s)$

$$\mu(I_{ij}) = \mu(I_i)\, \mu(I_j) \tag{23}$$

the authors of refs. 12 and 22 can concentrate a Gibbs measure ν_p on $E_F^{\varphi'(p)}$, i.e., $\nu_p(I_i) \simeq (\mu(I_i))^p\, |I_i|^{-\varphi(p)}$ for any i, where φ is defined like in (36) (see ref. 27). This allows the computation of the spectrum $d(\alpha) = \dim(E_F^\alpha)$ and the extension of the validity of the multi-fractal formalism for nonlinear self-similar functions (21).

But multi-fractals in physics or image processing are not necessarily exactly self-similar, and might have different renormalization properties at different scales. Besides, in ref. 28, Ben Nasr showed that the existence of Gibbs measures is not necessary for the multi-fractal formalism for measures. He also showed that it suffices to use measures constructed according to Frostman's method. The comprehension of this result is actually the starting point which led us to this paper. The reason is that for the computation of the spectrum of singularities $d(\alpha)$ for our non-self-similar functions, we use such measures. This will be explained in more details in Section 5. Let us now describe in details the results of this paper.

2. MAIN RESULTS

For the sake of simplicity, we will assume that, $m=1$, $\Omega =]0,1[$ and $d=2$. The general case (i.e., $m=1$ or 2, Ω is a bounded open set of $\mathbb{R}^m$ and $d \geqslant 2$) is similar. Consider two sequences $(\lambda_1^n)_{n\geqslant 1}$ and $(\lambda_2^n)_{n\geqslant 1}$ of numbers and suppose that there exist A and B such that:

$$\forall n \in \mathbb{N}^*, \quad \forall i = 1, 2, \qquad 0 < A \leqslant |\lambda_i^n| \leqslant B < 1 \tag{24}$$

We replace assumption (16) by a weaker one:

$$T_j(I) \subset I \qquad \forall j = 1, 2 \tag{25}$$

We call quasi-self-similar function F a series of the form (21) in which we replace $\lambda_{i_1} \cdots \lambda_{i_n}$ by $\lambda_{i_1}^1 \cdots \lambda_{i_n}^n$. This means that we might have different renormalization properties at different scales, i.e.,

$$F(x) = \sum_{n=0}^{\infty} \sum_{i=(i_1,\dots,i_n)\in\{1,2\}^n} \lambda_{i_1}^1 \cdots \lambda_{i_n}^n \, g(T_i^{-1}(x)) \tag{26}$$

(here also by convention $g(T_i^{-1}(x)) = 0$ for $x \notin I_i$ and the term of the series associated to $n=0$ is $g(x)$).

Remark that the new function F is written as a superposition of "similar" structures at different scales, reminiscent of some possible modelization of turbulence. The expression of F looks also like a wavelet decomposition (except that g has no cancellation necessarily).

Contrary to the iterated self-similar equation (20) satisfied by the functions F of both refs. 12 and 22, our function F (given by the series (26)) verifies for any integer $N \in \mathbb{N}^*$ and $x \in \mathbb{R}$ the quasi-self-similar equation

$$\begin{aligned} F(x) = {} & \sum_{n=0}^{N-1} \sum_{i=(i_1,\dots,i_n)\in\{1,2\}^n} \lambda_{i_1}^1 \cdots \lambda_{i_n}^n \, g(T_i^{-1}(x)) \\ & + \sum_{i=(i_1,\dots,i_N)\in\{1,2\}^N} \lambda_{i_1}^1 \cdots \lambda_{i_N}^N \, F_N(T_i^{-1}(x)) \end{aligned} \tag{27}$$

with

$$F_N(y) = \sum_{n=0}^{\infty} \sum_{j=(j_1,\ldots,j_n)\in\{1,2\}^n} \lambda_{j_1}^{N+1} \cdots \lambda_{j_n}^{N+n} g(T_j^{-1}(y)) \tag{28}$$

Let us be more precise, for example, for $N=1$, the quasi-self-similar equation is

$$F(x) = \begin{cases} g(x)+\lambda_1^1 F_1(T_1^{-1}(x)) & \text{if} \quad x \in I_1 \\ g(x)+\lambda_2^1 F_1(T_2^{-1}(x)) & \text{if} \quad x \in I_2 \\ g(x) & \text{elsewhere} \end{cases} \tag{29}$$

with

$$F_1(y) = \sum_{n=0}^{\infty} \sum_{i=(i_1,\ldots,i_n)\in\{1,2\}^n} \lambda_{i_1}^{2} \cdots \lambda_{i_n}^{1+n} g(T_i^{-1}(y)) \tag{30}$$

In general F_1 is different from F. So, F is not self-similar as in (15).

If for all $x \in I$, F is $C^k(x)$ then the multi-fractal formalism for $\alpha < k$ does not have any interest, and this is the reason why the authors of refs. 12 and 22 assumed that there exists a point $x_0 \in \Omega$ such that $F \notin C^k(x_0)$. From the characterization of the Hölder regularity by the wavelet transform, this condition implies that there exists $a_n \to 0$, b_n and $C_n \to +\infty$ such that

$$|b_n - x_0| \leqslant a_n \qquad \text{and} \qquad |C_{a_n, b_n}(F)| \geqslant C_n a_n^k \tag{31}$$

Remark 2. Since our quasi-self-similar function satisfies equation (27) then for every N, if $|i| = N$ and $x_0 \in I_i$ then $F \notin C^k(x_0)$ if and only if $F_N \notin C^k(T_i^{-1}(x_0))$. So a natural assumption for our extension is that the sequences a_n, b_n and C_n should be uniform on N. This means that there exist $a_n \longrightarrow 0$, b_n and $C_n \longrightarrow +\infty$ such that

$$|b_n - x_0| \leqslant a_n \qquad \text{and} \qquad \forall N \quad |C_{a_n, b_n}(F_N)| \geqslant C_n a_n^k \tag{32}$$

i.e., the sequence $(F_N)_N$ is not uniformly in $C^k(x_0)$. This new condition is possible; for example let g be a wavelet supported in $[0, 1]$, $T_0(x) = \frac{1}{3}x$ and $T_2(x) = \frac{1}{3}x + \frac{2}{3}$, and

$$F(x) = \sum_{j=0}^{+\infty} \sum_{i=(i_1,\ldots,i_j)\in\{0,2\}^j} \lambda_{i_1}^1 \cdots \lambda_{i_j}^j g(3^j x - 3^{j-1} i_1 - \cdots - 3 i_{j-1} - i_j)$$

then for $x = \sum_{j=1}^{\infty} i_j(x)\, 3^{-j} \in [0, 1]$

$$F_N(x) = \sum_{j=0}^{+\infty} \lambda_{i_1(x)}^{1+N} \cdots \lambda_{i_j(x)}^{j+N} g(3^j x - 3^{j-1} i_1(x) - \cdots - i_j(x))$$

The discrete wavelet coefficients of F_N are $|C_{j,l}(F_N)| = |\lambda_{i_1}^{1+N} \cdots \lambda_{i_j}^{j+N}|$ for $l 3^{-j} = i_1 3^{-1} + \cdots + i_j 3^{-j}$. If $3^{-k} < |\lambda_p^n|$ $\forall n \in \mathbb{N}^*$ and $\forall p = 0, 2$, then the discrete form of condition (32) (i.e., $a_n = 3^{-n}$, $b_n = l 3^{-n}$) holds.

Outside K, the quasi-self-similar function F given in (26) is locally a finite sum of functions generated by g. Therefore F is C^k outside K. The regularity of F at each point x of K is obtained by estimating the size of the wavelet transform in a neighborhood of x. We will use both the special expression (26) of F and the iterated quasi-self-similar functional equation (27) to prove that the wavelet transform of F satisfies a similar "quasi-self-similar" functional equation, which will enable us to estimate the size of the wavelet transform. We will prove that if $b \in I_{(i_1, \dots, i_n)} +]-a, a[$, then the order of magnitude of $|C_{a,b}(F)|$ is $|\lambda_{i_1}^1 \cdots \lambda_{i_n}^n|$. This will imply that for $x \in K$

$$\text{if} \quad a(x) = \liminf_{n \to \infty} \frac{\log |\lambda_{i_1(x)}^1 \cdots \lambda_{i_n(x)}^n|}{\log |I_{(i_1(x), \dots, i_n(x))}|} < k, \qquad \text{then} \quad \alpha_F(x) = a(x) \tag{33}$$

This allows also the computation of $\eta(p)$.

For the computation of the spectrum of singularities, the construction of the Gibbs measures explained above fails because in general, $\forall l, p \in \mathbb{N}^*$, $|\lambda_{i_1}^1| \cdots |\lambda_{i_l}^l|\, |\lambda_{j_1}^{l+1}| \cdots |\lambda_{j_p}^{l+p}|$ has not the same order of magnitude as $|\lambda_{i_1}^1| \cdots |\lambda_{i_l}^l|\, |\lambda_{j_1}^1| \cdots |\lambda_{j_p}^p|$. So, unlike the case (23) of self-similar functions, the corresponding quantity $\mu(I_{ij})$ has not the same order of magnitude as $\mu(I_i)\, \mu(I_j)$. The only method to compute $d(\alpha)$ is to use some results established by Ben Nasr and al (see refs. 28 and 29). For x and y reals, define

$$\underline{K}(x, y) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_i |\lambda_i|^x\, |I_i|^{-y};\ K \subset \bigcup_i I_i \text{ and } |I_i| \leqslant \varepsilon \right\} \tag{34}$$

$$C_n(x, y) = \frac{1}{n} \log \sum_{|i| = n} |\lambda_i|^x\, |I_i|^{-y} \quad \text{and} \quad C(x, y) = \limsup_{n \to +\infty} C_n(x, y) \tag{35}$$

and

$$\varphi(x) = \sup\{y;\ C(x, y) < 0\} \tag{36}$$

Let $B_j = \{i: 2^{-j} \leqslant |I_i| < 2.2^{-j}\}$. For $i = (i_1, ..., i_n)$ we denote $|i| = n$, $\lambda_i = \lambda^1_{i_1} \cdots \lambda^n_{i_n}$,

$$\alpha_{\min} = \liminf_{j \to \infty} \inf_{i \in B_j} \frac{\log |\lambda_i|}{\log |I_i|} \qquad \text{and} \qquad \alpha_{\max} = \limsup_{j \to \infty} \sup_{i \in B_j} \frac{\log |\lambda_i|}{\log |I_i|} \tag{37}$$

Theorem 1. If $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$, then $d(\alpha) = -\infty$.

Suppose that $\underline{K}(p, \varphi(p)) > 0$. Let $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ such that $\alpha < k$. If φ is derivable at p and if $\alpha = \varphi'(p)$ then $d(\alpha) = \alpha p - \varphi(p) = \inf_x(\alpha x - \varphi(x))$. Furthermore, if $\varphi(p) + 1 < kp$ then $\eta(p) = \varphi(p) + 1$ and

$$d(\alpha) = \alpha p - \eta(p) + 1 = \inf_{q:\, \varphi(q)+1 < kq} (\alpha q - \eta(q) + 1)$$

In the next section, we use the Littlewood-characterization of the $C^\alpha(\mathbb{R}^m)$ norm to compute the global Hölder regularity of F (see Proposition 2). In Section 4, we determine the Hölder exponent $\alpha_F(x)$ (see Theorems 2–4). In Section 5, we compute the spectrum of singularities (see Theorem 5). In Section 6, we determine the Besov exponent $\eta(q)$ and we prove the validity of the multi-fractal formalism (see Theorem 6). In Section 7, we prove that the box dimension of the graph of F is equal to $\sup(1, 1-\varphi(1))$. Finally, Section 8 is a "summary and prospects" section in which we apply our results for some examples of self-similar functions and quasi-self-similar functions. We give some examples of quasi-self-similar functions that are not self-similar and for which the function $\varphi(p)$ can be numerically estimated. We also give some applications of our results for representation of signals. Finally, we present some results concerning random quasi-self-similar functions.

3. GLOBAL HÖLDER REGULARITY

We will use the Littlewood-characterization of the $C^\alpha(\mathbb{R})$ norm to compute the global Hölder regularity of F (see refs. 13 and 30). Let ψ be a function in the Schwartz class such that

$$\hat{\psi}(\xi) = 0 \quad \text{for } |\xi| \leqslant 1 \qquad \text{and} \qquad |\xi| \geqslant 8$$

and

$$\hat{\psi}(\xi) = 1 \qquad \text{for} \quad 2 \leqslant |\xi| \leqslant 4$$

Let $\psi_l(x) = 2^l \psi(2^l x)$. Recall that a function F is $C^\alpha(\mathbb{R})$ if and only if there exists a constant $C \geqslant 0$ such that

$$|F * \psi_l(x)| \leqslant C 2^{-\alpha l} \qquad \forall x \in \mathbb{R}$$

Lemma 3. We have

$$0 < \alpha_{\min} \leqslant \alpha_{\max} < \infty \tag{38}$$

Proof of Lemma 3. If $i = (i_1, \ldots, i_n) \in B_j$ then $2^{-j} \leqslant |I_i| < 2^{-j+1}$. The relation (18) satisfied by T_1 and T_2 yields

$$\theta^n \leqslant |I_i| \leqslant \rho^n \tag{39}$$

Hence

$$-(j-1)\frac{\log 2}{\log \theta} < n \leqslant -j \frac{\log 2}{\log \rho}$$

Thus

$$\frac{(j-1)}{j}\frac{\log B}{\log \theta} < \frac{\log |\lambda_i|}{\log |I_i|} < \frac{j}{(j-1)}\frac{\log A}{\log \rho}$$

Hence

$$0 < \frac{\log B}{\log \theta} \leqslant \alpha_{\min} \leqslant \alpha_{\max} \leqslant \frac{\log A}{\log \rho} < \infty$$

Whence Lemma 3. ∎

Proposition 2. If $\alpha_{\min} \leqslant k$, then $F \in C^{\alpha_{\min} - \varepsilon}(\mathbb{R})$, $\forall \varepsilon > 0$.

Proof of Proposition 2. We split F as a sum

$$F(x) = \sum_{j \geqslant 0} \tilde{F}_j(x) \qquad \text{where} \quad \tilde{F}_j(x) = \sum_{i \in B_j} \lambda_i\, g(T_i^{-1}(x))$$

Let $\omega_{l,j} = \tilde{F}_j * \psi_l$ and $h_{i,l} = (g \circ T_i^{-1}) * \psi_l$.
We have

$$|h_{i,l}(x)| = \left| \int g(T_i^{-1}(y))\, \psi_l(x-y)\, dy \right|$$

Denote by $P_k g_x(h)$ the Taylor expansion of order k of g at x, i.e.,

$$P_k g_x(h) = \sum_{q \leqslant k} \frac{g^{(q)}(x)}{q!} h^q \tag{40}$$

Since g is supported in I and ψ is well localized and has enough vanishing moments, then for $x \in I_i$

$$|h_{i,l}(x)| = 2^l \left| \int_{I_i} (g(T_i^{-1}(y)) - P_{k-1}(g \circ T_i^{-1})_x (y-x)) \, \psi(2^l(x-y)) \, dy \right|$$

Using the Taylor theorem and the fact that g is a C^k function, we obtain

$$|h_{i,l}(x)| \leqslant 2^l \int |\psi(2^l(x-y))| \, (\sup_{u \in [x, y]} |(g \circ T_i^{-1})^{(k)} (u)|) \, |x-y|^k \, dy$$

The following result (cf. ref. 22) allows us to bound $\sup_{u \in [x, y]} |(g \circ T_i^{-1})^{(k)} (u)|$ by $C \, |I_i|^{-k}$:

Lemma 4. There exists a constant $C > 0$ such that

$$|T_i^{(l)}(x)| \leqslant C \, |I_i| \quad \forall |i| = n, \quad n \in \mathbb{N}^*, \quad l = 2,..., k+1 \quad \text{and} \quad x \in I$$

and

$$|(T_i^{-1})^{(l)}(x)| \leqslant C \, |I_i|^{-l} \quad \forall |i| = n, \quad n \in \mathbb{N}^*, \quad l = 2,..., k+1 \quad \text{and} \quad x \in I_i$$

Lemma 1 implies that for $i \in B_j$ and $u \in I_i$

$$\mathscr{D}^{-1} 2^{j-1} \leqslant \mathscr{D}^{-1} \, |I_i|^{-1} \leqslant |(T_i^{-1})' \, (u)| \leqslant \mathscr{D} \, |I_i|^{-1} \leqslant \mathscr{D} 2^j \tag{41}$$

Thanks to the second part of Lemma 4 and the fact that g is C^k and supported in I, we get

$$|(g \circ T_i^{-1})^{(k)} \, (u)| \leqslant C \, |I_i|^{-k} \leqslant C \, 2^{kj} \tag{42}$$

Hence

$$|h_{i,l}(x)| \leqslant C \, 2^{kj} 2^l \int |\psi(2^l(x-y))| \, |x-y|^k \, dy$$

Thus for $j \leqslant l$

$$|h_{i,l}(x)| \leqslant C 2^{kj} 2^{-kl}$$

Whence, for $j \leqslant l$

$$\sum_{i \in B_j : x \in I_i} |\lambda_i| \, |h_{i,l}(x)| \leqslant C \sum_{i \in B_j : x \in I_i} |\lambda_i| \, 2^{k(j-l)}$$

Lemma 5 (cf. refs. 12 and 22). Let $x_i = T_i(0)$ and L large enough. Set

$$B_{j,L}(x) = \{i \in B_j : |x - x_i| \leqslant L 2^{-j}\}$$

The cardinality of $B_{j,L}(x)$ is bounded independently of x and j by CL.

From Lemma 5 and the fact that for $x \in I_i$, $|x - x_i| \leqslant |I_i| \leqslant C 2^{-j}$, we obtain for $j \leqslant l$

$$\begin{aligned}\sum_{i \in B_j : x \in I_i} |\lambda_i| \, |h_{i,l}(x)| &\leqslant C (\sup_{i \in B_j} |\lambda_i|) \, 2^{k(j-l)} \\ &\leqslant C 2^{(-\alpha_{\min} + \varepsilon) j} 2^{k(j-l)}\end{aligned}$$

As a consequence

$$\begin{aligned}\sum_{0 \leqslant j \leqslant l} \sum_{i \in B_j : x \in I_i} |\lambda_i| \, |h_{i,l}(x)| &\leqslant C 2^{-kl} \sum_{0 \leqslant j \leqslant l} 2^{(-\alpha_{\min} + \varepsilon + k) j} \\ &\leqslant C 2^{(-\alpha_{\min} + \varepsilon) l}\end{aligned}$$

Now, if $x \notin I_i$, the localization and cancellation of ψ imply that $\psi = \Psi^{(k+1)}$, where Ψ has fast decay. Therefore

$$\begin{aligned}|h_{i,l}(x)| &= 2^l 2^{-(k+1) l} \left| \int_{I_i} (g \circ T_i^{-1})^{(k+1)} (y) \, \Psi(2^l (x - y)) \, dy \right| \\ &\leqslant 2^l 2^{-(k+1) l} \int_{I_i} 2^{(k+1) j} \frac{C_N}{(1 + 2^l \, |x - y|)^N} \, dy\end{aligned}$$

Since $|x - y| \geqslant \operatorname{dist}(x, I_i)$, then for $j \leqslant l$

$$|h_{i,l}(x)| \leqslant C_N \frac{2^{-kl} 2^{jk}}{(1 + 2^j \operatorname{dist}(x, I_i))^N}$$

If $i \in B_j$ and $|x - x_i| \leqslant L2^{-j}$ then $i \in B_{j,L}(x)$. In view of Lemma 5, we have

$$\sum_{i \in B_{j,L}(x)} \frac{1}{(1+2^j \operatorname{dist}(x, I_i))^N} \leqslant \operatorname{Card} B_{j,L}(x) \leqslant CL \tag{43}$$

On the other hand, if $i \in B_j$ and $|x - x_i| > L2^{-j}$ then there exists an integer $n \geqslant 1$ such that $nL2^{-j} \leqslant |x - x_i| < (n+1)\, L2^{-j}$. It follows that $i \in B_{j,(n+1)L}(x)$. Therefore

$$\sum_{i \in B_j : |x - x_i| > L2^{-j}} \frac{1}{(1+2^j \operatorname{dist}(x, I_i))^N} \leqslant \sum_{n \in \mathbb{N}^*} \sum_{i \in B_{j,(n+1)L}(x)} \frac{1}{(1+2^j \operatorname{dist}(x, I_i))^N} \tag{44}$$

Using Lemma 5 for $B_{j,(n+1)L}(x)$, we obtain

$$\sum_{n \in \mathbb{N}^*} \sum_{i \in B_{j,(n+1)L}(x)} \frac{1}{(1+2^j \operatorname{dist}(x, I_i))^N} \leqslant \sum_{n \in \mathbb{N}^*} C(n+1)\, L \frac{1}{(1+nL)^N} \tag{45}$$

The inequalities (43) and (45) imply that for $N > 2$

$$\sum_{i \in B_j} \frac{1}{(1+2^j \operatorname{dist}(x, I_i))^N} \leqslant C$$

Hence as previously, we obtain

$$\sum_{0 \leqslant j \leqslant l} \sum_{i \in B_j : x \notin I_i} |\lambda_i|\, |h_{i,l}(x)| \leqslant C\, 2^{(-\alpha_{\min} + \varepsilon)\, l}$$

On the other hand, for $j > l$, we have for any $x \in \mathbb{R}$

$$\begin{aligned} |\omega_{l,j}(x)| &\leqslant C \sup |\tilde{F}_j(x)| \\ &\leqslant C \sup_{i \in B_j} |\lambda_i| \\ &\leqslant C\, 2^{(-\alpha_{\min} + \varepsilon)\, j} \end{aligned}$$

Hence

$$\sum_{j > l} |\omega_{l,j}(x)| \leqslant C\, 2^{(-\alpha_{\min} + \varepsilon)\, l}$$

Whence for any $\varepsilon > 0$ and $x \in \mathbb{R}$

$$|F * \psi_l(x)| \leqslant C 2^{-(\alpha_{\min} - \varepsilon) l} \tag{46}$$

As a result, $F \in C^{\alpha_{\min} - \varepsilon}(\mathbb{R})$ for any $\varepsilon > 0$. ▮

4. THE POINTWISE HÖLDER EXPONENT

We will now compute the Hölder exponent of F at every point x.

Remark 3. It suffices to do this for $x \in K$, because outside K, F is locally a finite sum of functions generated by g, and g is C^k.

We divide the computation into two steps. The upper bound for $\alpha_F(x)$ will be obtained in Section 4.1, and the lower bound will be obtained in Section 4.2.

4.1. Upper Bound for Pointwise Hölder Exponent

We will use both the special expression (26) of F and the iterated quasi-self-similar functional equation (27) to prove that the wavelet transform of F satisfies a similar "quasi-self-similar" functional equation, which will enable us to estimate the size of the wavelet transform. We will prove that if $b \in I_{(i_1, \ldots, i_n)} +]-a, a[$, then the order of magnitude of $|C_{a,b}(F)|$ is $|\lambda^1_{i_1} \cdots \lambda^n_{i_n}|$. In view of the wavelet characterization (3) this will imply the following theorem.

Theorem 2. Let $x \in K$ and define

$$a(x) = \liminf_{n \to \infty} \frac{\log |\lambda^1_{i_1(x)} \cdots \lambda^n_{i_n(x)}|}{\log |I_{i_1(x) \cdots i_n(x)}|}$$

If $a(x) < k$ then

$$\alpha_F(x) \leqslant a(x) \tag{47}$$

Proof of Theorem 2. We will first estimate the size of the wavelet transform near x. Set $B_j(x) = B_{j,L}(x)$ and define

$$\Lambda_j(x) = \sup_{i \in B_j(x)} |\lambda_i|$$

and

$$\tilde{\Lambda}_j(x) = \sum_{l=1}^{j} \Lambda_l(x)\, 2^{-\hat{A}(j-l)} \qquad \text{with} \quad \hat{A} > \alpha_{\max}$$

Clearly

$$a(x) = \liminf_{j \to \infty} \frac{\log \Lambda_j(x)}{-j \log 2} = \liminf_{j \to \infty} \frac{\log \tilde{\Lambda}_j(x)}{-j \log 2} \tag{48}$$

Similarly to (27), we have for any $J \in \mathbb{N}^*$

$$F(x) = \sum_{j=0}^{J-1} \sum_{i \in B_j} \lambda_i\, g(T_i^{-1}(x)) + \sum_{i \in B_J} \lambda_i F_J(T_i^{-1}(x)) \tag{49}$$

with

$$F_J(y) = \sum_{n=0}^{\infty} \sum_{i \in \bigcup_{p \in \Delta_n} B_p} \lambda_{i_1}^{J+1} \cdots \lambda_{i_n}^{J+n}\, g(T_i^{-1}(y))$$

where $\Delta_n \subset \{n-3-d, n+2+d\}$ and d the unique integer such that $2^d \leqslant \mathscr{D} < 2.2^d$ (where $\mathscr{D}$ is the constant in the distortion Lemma). In fact,

$$F(x) = \sum_{j=0}^{J-1} \sum_{i \in B_j} \lambda_i\, g(T_i^{-1}(x)) + \sum_{j=J}^{\infty} \sum_{i \in B_j} \lambda_i\, g(T_i^{-1}(x))$$

and if $i \in B_j$ then $2^{-j} \leqslant |I_i| < 2.2^{-j}$, write $i = (l, m)$ with $l \in B_J$, therefore

$$2^{-(2+d+j-J)} \leqslant |I_m| < 2^{-(-2-d+j-J)}$$

Remark that F_J is bounded independently of J because the cardinality of Δ_n is at most $6+2d$. Using the necessary condition (3) in Proposition 1, it suffices to show the following result in order to prove Theorem 2.

Proposition 3. Let $x \in K$, $J \in \mathbb{N}$ large enough such that $\Lambda_J(x) \geqslant \frac{1}{2} \tilde{\Lambda}_J(x)$. There exists a branch $j^0 = (j_1^0, \ldots, j_n^0)$ in $B_J(x)$, $b \in I_{j^0}$ and $a \sim 2^{-J}$ such that

$$|b - x| \leqslant Ca \qquad \text{and} \qquad |C_{a,b}(F)| \geqslant C\Lambda_J(x)$$

Proof of Proposition 3. From the functional equation (49) satisfied by F, we get

$$C_{a,b}(F) = \sum_{j=0}^{J-1} \sum_{i \in B_j} \lambda_i \int_{I_i} \bar{\psi}_{a,b}(t)\, g(T_i^{-1}(t))\, dt + \sum_{i \in B_J} \lambda_i \int_{I_i} \bar{\psi}_{a,b}(t)\, F_J(T_i^{-1}(t))\, dt$$

By a change of variable, we obtain

$$C_{a,b}(F) = \sum_{j=0}^{J-1} \sum_{i \in B_j} \lambda_i \int_I \bar{\psi}_{a,b}(T_i(t))\, g(t)\, T_i'(t)\, dt \tag{50}$$

$$+ \sum_{i \in B_J} \lambda_i \int_I \bar{\psi}_{a,b}(T_i(t))\, F_J(t)\, T_i'(t)\, dt \tag{51}$$

In ref. 22, the author gave asymptotic developments for the composition of a wavelet by a contraction T_i. These developments are well adapted with the wavelet analysis in the sense that at small scales the action of such contraction on the wavelet is close to a translation and rescaling of a possibly different wavelet. Let us recall these developments.

Lemma 6 (cf. ref. 22). Let ψ be a compactly supported wavelet with enough smoothness and vanishing moments. Suppose that ψ is real and even. Let $b \in I_i$ and a such that $0 < a < |I_i|$, then

$$\psi_{a,b}(T_i(t)) = |(T_i^{-1})'\,(b)|\, \psi_{a\,|(T_i^{-1})'\,(b)|,\, T_i^{-1}(b)}(t)$$

$$+ \sum_{p=1}^{k-1} \sum_{l=2p}^{k-1+p} a^{l-p} A_i^{(p,l)}(b)\, \psi^{(p,l)}_{a\,|(T_i^{-1})'\,(b)|,\, T_i^{-1}(b)}(t) + (R_{a,b}^{i,k})(t) \tag{52}$$

where $\psi^{(p,l)}(t) = t^l \psi^{(p)}(t)$ is a compactly supported wavelet,

$$A_i^{(p,l)}(b) = \frac{1}{p!}\, |(T_i^{-1})'\,(b)|^{l+1} \sum_{\substack{2 \leqslant q_1,\ldots,q_p \leqslant k \\ q_1 + \cdots + q_p = l}} \prod_{m=1}^{p} \frac{T_i^{(q_m)}(T_i^{-1}(b))}{q_m!}$$

$$|A_i^{(p,l)}(b)| \leqslant C\, |I_i|^{p-l-1}$$

and $(R_{a,b}^{i,k})$ is a function supported in $|t - T_i^{-1}(b)| \leqslant Ca\, |I_i|^{-1}$ such that

$$|(R_{a,b}^{i,k})(t)| \leqslant Ca^{k-1}\, |I_i|^{-k} \qquad \forall t \tag{53}$$

and

$$\|(R_{a,b}^{i,k})(t)\|_{L^1(\mathbb{R})} \leqslant Ca^k\, |I_i|^{-k-1} \tag{54}$$

The previous lemma allows the estimation of the size of the wavelet transform near each point of K. Let j^0 be a branch of $B_J(x)$ for which $\Lambda_J(x) = \sup_{i \in B_J(x)} |\lambda_i|$ is reached (j^0 exists because of Lemma 5).

The assumption (32) taken in Remark 2 together with Remark 3 imply that

$$(b_n)_n \subset K +]-\varepsilon, \varepsilon[\tag{55}$$

where ε is a constant small enough. Take $b = T_{j^0}(b_n)$ for n large enough and $a = a_n\, |T'_{j^0}(b_n)|$, then

$$b \in I_{j^0}, \qquad a \sim a_n\, |I_{j^0}| \sim 2^{-J}$$

and

$$|x - b| \leqslant |x - x_{j^0}| + |x_{j^0} - b| \leqslant L2^{-J} + |I_{j^0}| \leqslant C\,2^{-J} \leqslant Ca$$

Remark 4.

$$|b_n - x_0| \leqslant a_n \qquad \text{implies that} \quad |b - T_{j^0}(x_0)| \leqslant Ca_n\, |I_{j^0}| \tag{56}$$

Since $x_0 \in K$ then $T_{j^0}(x_0) \in K$. Relations (55) and (56) imply that b is near both K and $T_{j^0}(x_0)$.

Remark 5. If H_i denotes a "hole" in I_i, then by argument similar to the one given in the proof of Lemma 1, we get (see refs. 23 and 24)

$$\mathscr{D}^{-1}\, |H_i| \leqslant |T'_i(x)| \leqslant \mathscr{D}\, |H_i| \quad \forall x \in I\,, |i| = n \quad \text{and} \quad n \in \mathbb{N}^*$$

Thus, Lemma 1 implies that

$$|H_i| \geqslant \mathscr{D}^{-2}\, |I_i| \qquad \forall |i| = n \quad \text{and} \quad n \in \mathbb{N}^* \tag{57}$$

- If "$0 \leqslant j \leqslant J-1$, $i \in B_j$ and $b \notin I_i$" or "$i \in B_J$, $b \notin I_i$ and $|i| \leqslant |j^0|$" then, from Remark 5 and the fact that $b \in I_{j^0}$, we have

$$\operatorname{dist}(b, I_i) \geqslant |H_i| \geqslant C\, |I_i| \geqslant C\, |I_{j^0}| \tag{58}$$

We deduce that $|T_i(t) - b| \geqslant C\, |I_{j^0}|$ for all $t \in I$, and since $a \sim a_n\, |I_{j^0}|$ and ψ has compact support then taking a_n small enough the corresponding integrals in (50) and (51) vanish.

- If $i \in B_J$, $b \notin I_i$ and $i = (j^0, 2, 2)$ or $i = (j^0, 1, 1)$ then

$$\operatorname{dist}(b, I_i) \geqslant |H_i| \geqslant C\, |I_i| \geqslant C\, |I_{j^0}| \tag{59}$$

Hence the corresponding integrals in (51) vanish.

- If $i \in B_J$, $b \notin I_i$ and $i=(j^0, 2, 1, \ldots, 1, 2)$ (resp. $i=(j^0, 1, 2, \ldots 2, 1)$) then

$$\operatorname{dist}(b, I_i) \geqslant |H_{(j^0, 2, 1, \ldots, 1)}| \geqslant C\, |I_{(j^0, 2, 1, \ldots, 1)}| \tag{60}$$

$$(\text{resp.} \quad \operatorname{dist}(b, I_i) \geqslant |H_{(j^0, 1, 2, \ldots, 2)}| \geqslant C\, |I_{(j^0, 1, 2, \ldots, 2)}|) \tag{61}$$

In view of Lemma 2 and the fact that $i \in B_J$ and $j^0 \in B_J$, the length of the stretch of ones (resp. 2's) is independently bounded, so $|I_{(j^0, 2, 1, \ldots, 1)}| \geqslant C\, |I_{j^0}|$ (resp. $|I_{(j^0, 1, 2, \ldots, 2)}| \geqslant C\, |I_{j^0}|$), we deduce that $|T_i(t)-b| \geqslant C\, |I_{j^0}|$ for all $t \in I$, and since $a \sim a_n\, |I_{j^0}|$ and ψ has compact support then taking a_n small enough the corresponding integrals in (51) vanish.

- If $i \in B_J$, $b \notin I_i$ and $i=(j^0, 2, 1, \ldots, 1)$ (resp. $i=(j^0, 1, 2, \ldots, 2)$) then

$$\left| \lambda_i \int_{I_i} \bar{\psi}_{a,b}(t)\, F_J(T_i^{-1}(t))\, dt \right| \leqslant C\, |\lambda_{j^0}| \leqslant C\Lambda_J(x) \tag{62}$$

As above, the length of the stretch of ones (resp. 2's) is independently bounded. So, the contribution of all these branches is $O(\Lambda_J(x))$.

Now, remark that if $i \in B_j$ (resp. $i \in B_J$) and $b \in I_i$ then $i \in B_j(b)$ (resp. $i \in B_J(b)$).

Let $C^{(p,l)}_{a,b}(F)$ be the $\psi^{(p,l)}$-wavelet transform of F. Using (50), (51) and Lemma 6, the ψ-wavelet transform of F will satisfy the following ("wavelet quasi-self-similar") relation

$$C_{a,b}(F) = O(\Lambda_J(x)) \tag{63}$$

$$+ \sum_{j=0}^{J-1} \sum_{i \in B_j(b)} \lambda_i\, |(T_i^{-1})'(b)|\, C_{a|(T_i^{-1})'(b)|, T_i^{-1}(b)}(gT'_i) \tag{64}$$

$$+ \sum_{i \in B_J(b)} \lambda_i\, |(T_i^{-1})'(b)|\, C_{a|(T_i^{-1})'(b)|, T_i^{-1}(b)}(F_J T'_i) \tag{65}$$

$$+ \sum_{p=1}^{k-1} \sum_{l=2p}^{k-1+p} a^{l-p} \sum_{j=0}^{J-1} \sum_{i \in B_j(b)} \lambda_i A_i^{(p,l)}(b)\, C^{(p,l)}_{a|(T_i^{-1})'(b)|, T_i^{-1}(b)}(gT'_i) \tag{66}$$

$$+ \sum_{p=1}^{k-1} \sum_{l=2p}^{k-1+p} a^{l-p} \sum_{i \in B_J(b)} \lambda_i A_i^{(p,l)}(b)\, C^{(p,l)}_{a|(T_i^{-1})'(b)|, T_i^{-1}(b)}(F_J T'_i) \tag{67}$$

$$+ \sum_{j=0}^{J-1} \sum_{i \in B_j(b)} \lambda_i \int (R^{i,k}_{a,b})(t)\, g(t)\, T'_i(t)\, dt \tag{68}$$

$$+ \sum_{i \in B_J(b)} \lambda_i \int (R^{i,k}_{a,b})(t)\, F_J(t)\, T'_i(t)\, dt \tag{69}$$

We will show that the wavelet transform of F is large near the branch j^0, and that the term (65) corresponding to the branch $i = j^0$ is the main term in the "wavelet quasi-self-similar" relation. We first prove that for n large enough

$$|\lambda_{j^0}| \, |(T_{j^0}^{-1})'(b)| \, |C_{a\,|(T_{j^0}^{-1})'(b)|,\, T_{j^0}^{-1}(b)}(F_J T'_{j^0})| \geqslant \frac{C_n}{2} \Lambda_J(x) \tag{70}$$

From (32) and the fact that $b = T_{j^0}(b_n)$ and $a = a_n \, |T'_{j^0}(b_n)|$, we have

$$|C_{a\,|(T_{j^0}^{-1})'(b)|,\, T_{j^0}^{-1}(b)}(F_J)| \geqslant C_n a^k \, |(T_{j^0}^{-1})'(b)|^k \tag{71}$$

On the other hand,

$$\begin{aligned}
&|(T_{j^0}^{-1})'(b)| \, |C_{a\,|(T_{j^0}^{-1})'(b)|,\, T_{j^0}^{-1}(b)}(F_J T'_{j^0})| \\
&\quad = \frac{1}{a} \left| \int \psi\left(\frac{t - T_{j^0}^{-1}(b)}{a (T_{j^0}^{-1})'(b)} \right) F_J(t) \, T'_{j^0}(t) \, dt \right| \\
&\quad = \frac{1}{a} \left| \int \psi\left(\frac{t - T_{j^0}^{-1}(b)}{a (T_{j^0}^{-1})'(b)} \right) F_J(t) \, (T'_{j^0}(T_{j^0}^{-1}(b)) + O^{(2)}_{j^0}(|t - T_{j^0}^{-1}(b)|)) \, dt \right|
\end{aligned}$$

with

$$\begin{aligned}
O^{(2)}_{j^0}(|t - T_{j^0}^{-1}(b)|) &\leqslant (\sup |T''_{j^0}(u)|) \, |t - T_{j^0}^{-1}(b)| \\
&\leqslant C \, |I_{j^0}| \, |t - T_{j^0}^{-1}(b)| \\
&\leqslant C |T'_{j^0}(T_{j^0}^{-1}(b))| \, |t - T_{j^0}^{-1}(b)|
\end{aligned}$$

Thus

$$\begin{aligned}
&|(T_{j^0}^{-1})'(b)| \, |C_{a\,|(T_{j^0}^{-1})'(b)|,\, T_{j^0}^{-1}(b)}(F_J T'_{j^0})| \\
&\quad \geqslant \frac{1}{a \, |(T_{j^0}^{-1})'(b)|} \left| \int \psi\left(\frac{t - T_{j^0}^{-1}(b)}{a (T_{j^0}^{-1})'(b)} \right) F_J(t) \, dt \right| \\
&\qquad - C \frac{1}{a \, |(T_{j^0}^{-1})'(b)|} \int \left| \psi\left(\frac{t - T_{j^0}^{-1}(b)}{a (T_{j^0}^{-1})'(b)} \right) \right| |F_J(t)| \, |t - T_{j^0}^{-1}(b)| \, dt
\end{aligned}$$

From (71) and the fact that F_J is bounded independently of J and $a \, |(T_{j^0}^{-1})'(b)| \sim 1$ it follows that

$$|(T_{j^0}^{-1})'(b)| \, |C_{a\,|(T_{j^0}^{-1})'(b)|,\, T_{j^0}^{-1}(b)}(F_J T'_{j^0})| \geqslant C_n a^k \, |(T_{j^0}^{-1})'(b)|^k - C$$

Hence, for n large enough (70) holds.

Consider now the term (65) for which we exclude the branch $i = j^0$. Since $a\,|(T_i^{-1})'\,(b)| \sim 1$ and F_J is bounded independently of J, then

$$|(T_i^{-1})'\,(b)|\;|C_{a\,|(T_i^{-1})'\,(b)|,\,T_i^{-1}(b)}(F_J T_i')| \leqslant C$$

From Lemma 5 and the fact that the branches i we consider are in $B_J(b)$, it follows that

$$\sum_{\substack{i \in B_J \\ i \neq j^0}} |\lambda_i|\;|(T_i^{-1})'\,(b)|\;|C_{a\,|(T_i^{-1})'\,(b)|,\,T_i^{-1}(b)}(F_J T_i')| \leqslant C\tilde{\Lambda}_J(b) \tag{72}$$

Now, let us estimate the righthand side of (64). We will prove that for $0 \leqslant j \leqslant J-1$ and $i \in B_j$ the terms $|(T_i^{-1})'\,(b)|\;|C_{a\,|(T_i^{-1})'\,(b)|,\,T_i^{-1}(b)}(gT_i')|$ decay like $(a\,|(T_i^{-1})'\,(b)|)^k$ because of the smoothness of g (i.e., $g \in C^k(\mathbb{R})$) and the cancellation of the wavelet. This will imply that the righthand side of (64) is bounded by $C\tilde{\Lambda}_J(b)$.

For $0 \leqslant j \leqslant J-1$ and $i \in B_j$, we have

$$\begin{aligned}
&|(T_i^{-1})'\,(b)|\;|C_{a\,|(T_i^{-1})'\,(b)|,\,T_i^{-1}(b)}(gT_i')| \\
&\quad = \frac{1}{a}\left|\int \psi\left(\frac{t-T_i^{-1}(b)}{a(T_i^{-1})'\,(b)}\right) g(t)\,T_i'(t)\,dt\right| \\
&\quad = \frac{1}{a}\left|\int \psi\left(\frac{t-T_i^{-1}(b)}{a(T_i^{-1})'\,(b)}\right) (g(t)\,T_i'(t) - P_{k-1}(gT_i')_{T_i^{-1}(b)}\,(t-T_i^{-1}(b)))\,dt\right|
\end{aligned}$$

Using the mean value theorem, the previous term will be bounded by

$$\frac{1}{a}\int \left|\psi\left(\frac{t-T_i^{-1}(b)}{a(T_i^{-1})'\,(b)}\right)\right| \Big(\sup_{u \in [t,\,T_i^{-1}(b)]} |(gT_i')^{(k)}\,(u)|\Big)\,|t-T_i^{-1}(b)|^k\,dt$$

From both formula

$$(gT_i')^{(k)}\,(u) = \sum_{q=0}^{k} C_k^q\,g^{(k-q)}(u)\,T_i^{(q+1)}(u)$$

Lemmas 1 and 4 and the fact that g has compact support, it follows that

$$|(gT_i')^{(k)}\,(u)| \leqslant C\,|I_i|$$

So $|(T_i^{-1})'\,(b)|\;|C_{a\,|(T_i^{-1})'\,(b)|,\,T_i^{-1}(b)}(gT_i')|$ is bounded by

$$\frac{C}{a\,|(T_i^{-1})'\,(b)|}\int \left|\psi\left(\frac{t-T_i^{-1}(b)}{a(T_i^{-1})'\,(b)}\right)\right| |t-T_i^{-1}(b)|^k\,dt \leqslant Ca^k 2^{kj}$$

Thus for $0 \leqslant j \leqslant J-1$

$$\sum_{i \in B_j} |\lambda_i| \, |(T_i^{-1})'(b)| \, |C_{a\,|(T_i^{-1})'(b)|,\, T_i^{-1}(b)}(gT'_i)| \leqslant C \sum_{i \in B_j(b)} |\lambda_i| \, a^k 2^{kj}$$

which by Lemma 5 will be bounded by $Ca^k 2^{kj} \tilde{\Lambda}_j(b)$.

Therefore, (48) and the assumption $a(x) < k$ imply that

$$\begin{aligned}\sum_{j \leqslant J-1} \sum_{i \in B_j} |\lambda_i| \, |(T_i^{-1})'(b)| \, |C_{a\,|(T_i^{-1})'(b)|,\, T_i^{-1}(b)}(gT'_i)| &\leqslant Ca^k \sum_{j \leqslant J-1} 2^{kj} \tilde{\Lambda}_j(b) \\ &\leqslant Ca^k 2^{kJ} \tilde{\Lambda}_J(b)\end{aligned}$$

Hence

$$\sum_{j \leqslant J-1} \sum_{i \in B_j} |\lambda_i| \, |(T_i^{-1})'(b)| \, |C_{a\,|(T_i^{-1})'(b)|,\, T_i^{-1}(b)}(gT'_i)| \leqslant C \tilde{\Lambda}_J(b) \tag{73}$$

Let us now estimate the terms (66) and (67). Thanks to the property $|A_i^{(p,l)}| \leqslant |I_i|^{p-l-1}$, the previous arguments give us

$$\begin{aligned}\sum_{p=1}^{k-1} \sum_{l=2p}^{k-1+p} a^{l-p} \sum_{\substack{i \in B_J \\ i \neq j^0}} |\lambda_i| \, |A_i^{(p,l)}(b)| \, |C^{(p,l)}_{a\,|(T_i^{-1})'(b)|,\, T_i^{-1}(b)}(F_J T'_i)| &\leqslant C \sum_{p=1}^{k-1} \sum_{l=2p}^{k-1+p} a^{l-p} \tilde{\Lambda}_J(b) \, a^{p-l-1} a \\ &\leqslant C \tilde{\Lambda}_J(b)\end{aligned}$$

and

$$\begin{aligned}&\sum_{p=1}^{k-1} \sum_{l=2p}^{k-1+p} a^{l-p} \sum_{j=0}^{J-1} \sum_{i \in B_j} |\lambda_i| \, |A_i^{(p,l)}(b)| \, |C^{(p,l)}_{a\,|(T_i^{-1})'(b)|,\, T_i^{-1}(b)}(gT'_i)| \\ &\leqslant C \sum_{p=1}^{k-1} \sum_{l=2p}^{k-1+p} a^{l-p} \sum_{j=0}^{J-1} \sum_{i \in B_j(b)} |\lambda_i| \, |I_i|^{p-l} \, |(T_i^{-1})'(b)| \, |C^{(p,l)}_{a\,|(T_i^{-1})'(b)|,\, T_i^{-1}(b)}(gT'_i)| \\ &\leqslant C \sum_{p=1}^{k-1} \sum_{l=2p}^{k-1+p} a^{l-p} \sum_{j=0}^{J-1} \tilde{\Lambda}_j(b) \, |I_i|^{p-l} \, a^k 2^{kj} \\ &\leqslant C \sum_{p=1}^{k-1} \sum_{l=2p}^{k-1+p} a^{k+l-p} \sum_{j=0}^{J-1} \tilde{\Lambda}_j(b) \, 2^{j(k+l-p)} \\ &\leqslant C \sum_{p=1}^{k-1} \sum_{l=2p}^{k-1+p} a^{k+l-p} \tilde{\Lambda}_J(b) \, 2^{J(k+l-p)} \\ &\leqslant C \tilde{\Lambda}_J(b)\end{aligned}$$

On the other hand, since F_J is bounded independently of J, then

$$\begin{aligned}\sum_{p=1}^{k-1}\sum_{l=2p}^{k-1+p} a^{l-p}\,|\lambda_{j^0}|\,|A_{j^0}^{(p,l)}(b)|\,|C^{(p,l)}_{a\,|(T_{j^0}^{-1})'(b)|,\,T_{j^0}^{-1}(b)}(F_J T'_{j^0})| &\leqslant C\sum_{p=1}^{k-1}\sum_{l=2p}^{k-1+p} a^{l-p}\Lambda_J(x)\,a^{p-l-1}a\\ &\leqslant C\Lambda_J(x)\end{aligned}$$

For the term (68), we bound it by

$$\sum_{j=0}^{J-1}\sum_{i\in B_j(b)}|\lambda_i|\int |(R^{i,k}_{a,b})(t)|\,|T'_i(t)|\,dt \leqslant C\sum_{j=0}^{J-1}\sum_{i\in B_j(b)}|\lambda_i|\,\|(R^{i,k}_{a,b})(t)\|_{L^1(\mathbb{R})}\,2^{-j}$$

which by (54) will be bounded by

$$C\sum_{j=0}^{J-1}\tilde{\Lambda}_j(b)\,a^k 2^{(k+1)j}2^{-j}\leqslant C\tilde{\Lambda}_J(b)$$

For the term (69), we use the fact that F_J is bounded independently of J to estimate it by

$$C\sum_{i\in B_J(b)}|\lambda_i|\,a^k 2^{(k+1)J}2^{-J}\leqslant C\tilde{\Lambda}_J(b)$$

Now, from the previous estimations, we get

$$|C_{a,b}(F)-\lambda_{j^0}\,|(T_{j^0}^{-1})'(b)|\,C_{a\,|(T_{j^0}^{-1})'(b)|,\,T_{j^0}^{-1}(b)}(F_J T'_{j^0})|\leqslant C\tilde{\Lambda}_J(b)$$

Choose C_n large enough, then (70) yields

$$|C_{a,b}(F)|\geqslant\tfrac{1}{2}\,|\lambda_{j^0}|\,|(T_{j^0}^{-1})'(b)|\,|C_{a\,|(T_{j^0}^{-1})'(b)|,\,T_{j^0}^{-1}(b)}(F_J T'_{j^0})| \tag{74}$$

Whence (70) and (74) yield Proposition 3. And thanks to (3), this proposition yields the upper bound for the Hölder exponent. ▮

4.2. Lower Bound for Pointwise Hölder Regularity

To obtain a lower bound for $\alpha_F(x)$, we will use definition (1). We separate two cases. In the first one (we denote it case 1), we assume that at least we have a middle "hole," i.e., $T_1(I)=[a_1,a_2]$ and $T_2(I)=[a_3,a_4]$ with $0\leqslant a_1<a_2<a_3<a_4\leqslant 1$. In the second case (we denote it case 2), we assume that we have three "holes," in the sense that, $T_1(I)=[a_1,a_2]$

and $T_2(I) = [a_3, a_4]$ with $0 < a_1 < a_2 < a_3 < a_4 < 1$. The main difference between the results of these cases is that in the first one we restrict the proof of the lower bound of the Hölder exponent on a (large) subset $\mathscr{R}$ of K (see below), which excludes points of K associated to very long but finite stretches of ones and 2's in their codes. Nevertheless, this exclusion does not affect the computation of the spectrum of singularities.

4.2.1. Case 1. $T_1(I)=[a_1, a_2]$ and $T_2(I)=[a_3, a_4]$ with $0 \leqslant a_1 < a_2 < a_3 < a_4 \leqslant 1$

Given $x \in [0, 1]$ and $N \in \mathbb{N}^*$, we define

$$L_N^1(x) = \begin{cases} 0 & \text{if } \ i_N(x) = 2 \\ L & \text{if } \ i_N(x) = i_{N+1}(x) = \cdots = i_{N+L-1}(x) = 1 \end{cases} \tag{75}$$

Clearly $L_N^1(x)$ is the length of the uninterrupted stretch of ones following (and including) $i_N(x)$. We define $L_N^2(x)$ analogously as the length of the stretch of 2's following $i_N(x)$.

We shall restrict ourselves to the set $\mathscr{R} := \{x \in K; \lim_{N \to +\infty} \frac{L_N^1(x)}{N} = \lim_{N \to +\infty} \frac{L_N^2(x)}{N} = 0\}$.

Theorem 3. If $x \in \mathscr{R}$ then $\alpha_F(x) \geqslant a(x)$.

Proof of Theorem 3. Using definition (1), we will prove the lower bound for the Hölder exponent $\alpha_F(x)$. Let $x \in K$. For $\beta < a(x)$, define

$$P_{[\beta]} F_x(h) = \sum_{j=0}^{\infty} \sum_{i \in B_j} \lambda_i P_{[\beta]} (g \circ T_i^{-1})_x \, (h)$$

as in (40). Thanks to (42)

$$|P_{[\beta]} F_x(h)| \leqslant \sum_{j=0}^{\infty} \tilde{\Lambda}_j(x) \, 2^{j[\beta]}$$

and since $\beta < a(x)$, then (48) implies that $P_{[\beta]} F_x(h)$ converges.

Let $i(n, x) = (i_1(x),..., i_n(x))$, J such that $2^{-J} \leqslant |h| < 2.2^{-J}$ and let N be the largest integer such that $x+h \in I_{i(N, x)}$. We have $|h| = |x+h-x| \leqslant |I_{i(N, x)}|$ and $x+h \notin I_{i(N+1, x)}$. So,

$$F(x+h) = \sum_{n=0}^{N} \lambda_{i(n, x)} \, g(T_{i(n, x)}^{-1}(x+h)) \qquad \text{and}$$

$$P_{[\beta]} F_x(h) = \sum_{n=0}^{\infty} \lambda_{i(n, x)} P_{[\beta]} (g \circ T_i^{-1})_x \, (h)$$

We have

$$F(x+h)-P_{[\beta]}F_x(h)=\sum_{n=0}^{N}\lambda_{i(n,x)}(g(T_i^{-1}(x+h))-P_{[\beta]}(g\circ T_i^{-1})_x(h)) \tag{76}$$

$$-\sum_{n\geqslant N+1}\lambda_{i(n,x)}P_{[\beta]}(g\circ T_i^{-1})_x(h) \tag{77}$$

The first term (76) is bounded by

$$C\,|h|^{[\beta]+1}\sum_{n=0}^{N}|\lambda_{i(n,x)}|\,|I_{i(n,x)}|^{-([\beta]+1)} \tag{78}$$

Let n_0 such that $|\lambda_{i(n,x)}|\leqslant|I_{i(n,x)}|^{(a(x)-\varepsilon)}$ $\forall n\geqslant n_0$, then (78) is bounded by

$$C\,|h|^{[\beta]+1}\sum_{n=0}^{n_0}|\lambda_{i(n,x)}|\,|I_{i(n,x)}|^{-([\beta]+1)}+\sum_{n=n_0}^{N}|I_{i(n,x)}|^{a(x)-\varepsilon-([\beta]+1)}\qquad\text{if}\quad N\geqslant n_0 \tag{79}$$

$$C\,|h|^{[\beta]+1}\sum_{n=0}^{N}|\lambda_{i(n,x)}|\,|I_{i(n,x)}|^{-([\beta]+1)}\qquad\text{if}\quad N\leqslant n_0 \tag{80}$$

The term (80) (resp. (79)) is bounded by $C\,|h|^{\beta}$ (resp. $C\,|h|^{[\beta]+1}(1+\sum_{n=n_0}^{N}|I_{i(n,x)}|^{\beta-([\beta]+1)})$).

But for $\varepsilon'=-\beta+[\beta]+1$, in view of Lemma 2, we have

$$\mathscr{D}^{-\varepsilon'}\,|I_{i(n,x)}|^{-\varepsilon'}\,|I_{i_{n+1}}|^{-\varepsilon'}\leqslant|I_{i(n+1,x)}|^{-\varepsilon'}\leqslant\mathscr{D}^{\varepsilon'}\,|I_{i(n,x)}|^{-\varepsilon'}\,|I_{i_{n+1}}|^{-\varepsilon'}$$

Without any loss of generality, we can assume that $h<0$. Thus from the definition of N, $i_{N+1}(x)=2$.

If $i_{N+1}(x+h)=1$ then $|h|\geqslant|H_{i(N,x)}|\geqslant C\,|I_{i(N,x)}|$.

If $i_{N+2}(x)=1=i_{N+2}(x)=\cdots=i_{N+L^1_{N+2}(x)}(x)$ and $i_{N+L^1_{N+2}(x)+1}(x)=2$.

Using (57) we obtain, $|h|\geqslant|H_{i(N+L^1_{N+2}(x),x)}|\geqslant C\,|I_{i(N+L^1_{N+2}(x),x)}|\geqslant C\mathscr{D}^{-L^1_{N+2}(x)}\,|I_{i(N,x)}|$.

Since $x\in\mathscr{R}$, then $\lim_{N\to+\infty}\frac{L^1_{N+2}(x)}{N}=0$. Let $\varepsilon>0$, there exists N_0 such that for all $N\geqslant N_0$, $L^1_{N+2}(x)\leqslant\varepsilon N$. Hence

$$|h|\geqslant C\mathscr{D}^{-\varepsilon N}\,|I_{i(N,x)}| \tag{81}$$

Hence, if J' is such that $2^{-J'}\leqslant|I_{i(N,x)}|<2.2^{-J'}$, then $|J'-J|\leqslant C$, with C a constant. Now, we can write

$$\sum_{n\leqslant N}|I_{i(n,x)}|^{-\varepsilon'}\sim\sum_{j\leqslant J'}\sum_{i\in B_j(x)}|I_i|^{-\varepsilon'}\leqslant C\sum_{j\leqslant J'}2^{j\varepsilon'}\leqslant C\,2^{J'\varepsilon'}$$

Thus, if $N \geqslant 0$ then

$$C\,|h|^{[\beta]+1}\left(1+\sum_{n=n_0}^{N}|I_{i(n,x)}|^{\beta-([\beta]+1)}\right)\leqslant C\,|h|^{[\beta]+1}\,(1+2^{-J'(\beta-[\beta]-1)})$$
$$\leqslant C\,|h|^{\beta}+C\,|h|^{[\beta]+1}\,2^{-J'(\beta-[\beta]-1)}$$
$$\leqslant C\,|h|^{\beta}+C\,2^{(J'-J)([\beta]+1)}2^{-J'\beta}$$
$$\leqslant C\,|h|^{\beta}+C\,|I_{i(N,x)}|^{\beta}$$
$$\leqslant C\,|h|^{\beta-\varepsilon}\qquad\text{(thanks to (81))}$$

On the other hand, for the term (77), we have

$$\sum_{j>J'}\sum_{i\in B_j(x)}|\lambda_i|\sum_{q=0}^{[\beta]}|I_i|^{-q}\,|h|^q\leqslant C\sum_{j>J'}2^{-\beta j}\left(\sum_{q=0}^{[\beta]}2^{qj}2^{-qJ'}\right)$$
$$\leqslant C\sum_{j>J'}2^{-\beta j}2^{[\beta]j}2^{-[\beta]J'}$$
$$\leqslant C\,2^{-\beta J'}$$
$$\leqslant C\,|I_{i(N,x)}|^{\beta}$$
$$\leqslant C\,|h|^{\beta-\varepsilon}$$

Consequently

$$|F(x+h)-P_{[\beta]}F_x(h)|\leqslant C\,|h|^{\beta-\varepsilon}\tag{82}$$

Hence Theorem 3. ∎

4.2.2. Case 2. $T_1(I)=[a_1, a_2]$ and $T_2(I)=[a_3, a_4]$ with $0<a_1<a_2<a_3<a_4<1$

Theorem 4. If $x\in K$ then $\alpha_F(x)\geqslant a(x)$.

Denote by $(l-H)_i$, $(m-H)_i$ and $(r-H)_i$ respectively the left, middle, and right "holes" in I_i, i.e.,

$$I_i=(l-H)_i\cup I_{i,1}\cup(m-H)_i\cup I_{i,2}\cup(r-H)_i$$

Relation (57) implies that

$$|(v-H)_i|\geqslant\mathscr{D}^{-2}\,|I_i|\quad\forall v\in\{l,m,r\},\quad|i|=n\quad\text{and}\quad n\in\mathbb{N}^*\tag{83}$$

We have $|h| = |x+h-x| \leqslant |I_{(i_1(x),\dots,i_N(x))}|$ and $x+h \notin I_{(i_1(x),\dots,i_N(x),i_{N+1}(x))}$. Here

$$|h| \geqslant |(l-H)_{(i_1(x),\dots,i_{N+1}(x))}| \qquad \text{if} \quad h<0$$

and

$$|h| \geqslant |(r-H)_{(i_1(x),\dots,i_{N+1}(x))}| \qquad \text{if} \quad h>0$$

So, we obtain

$$\begin{aligned}|h| &\geqslant \mathscr{D}^{-2}\, |I_{(i_1(x),\dots,i_N(x),i_{N+1}(x))}| \\ &\geqslant \mathscr{D}^{-2}\mathscr{D}^{-1}\, |I_{(i_1(x),\dots,i_N(x))}|\, |I_{i_{N+1}(x)}| \\ &\geqslant \mathscr{D}^{-3}\theta\, |I_{(i_1(x),\dots,i_N(x))}|\end{aligned}$$

Hence if J' is such that $2^{-J'} \leqslant |I_{(i_1(x),\dots,i_N(x))}| < 2.2^{-J'}$, then $|J'-J| \leqslant C$, with C a constant. Now, we can write

$$\begin{aligned}F(x+h) - P_{[\beta]}F_x(h) = &\sum_{j<J'} \sum_{\{i\in B_j\,:\,x\in I_i\}} \lambda_i \big(g(T_i^{-1}(x+h)) - P_{[\beta]}(g\circ T_i^{-1})_x\,(h)\big) \\ &+ \sum_{\{i\in B_{J'}\,:\,x\in I_i\}} \lambda_i F_{J'}(T_i^{-1}(x+h)) \\ &- \sum_{j\geqslant J'} \sum_{i\in B_j} \lambda_i P_{[\beta]}(g\circ T_i^{-1})_x\,(h)\end{aligned}$$

For each j of the series of the first term, there is a finite terms, thus if j_0 is such that $\tilde{\Lambda}_j(x) \leqslant 2^{-j(\alpha(x)-\epsilon)}$ for all $j \geqslant j_0$, then using the mean value theorem and (42), the first term will be bounded by

$$\begin{aligned}&C\sum_{j<J'} \tilde{\Lambda}_j(x)\, |h|^{[\beta]+1}\, 2^{j([\beta]+1)} \\ &\quad\leqslant C\,|h|^{[\beta]+1} \sum_{j<j_0} \tilde{\Lambda}_j(x)\, 2^{j([\beta]+1)} + C\,|h|^{[\beta]+1} \sum_{j_0\leqslant j<J'} 2^{-\beta j} 2^{j([\beta]+1)} \\ &\quad\leqslant C\,|h|^{[\beta]+1} + C\,|h|^{[\beta]+1}\, 2^{-\beta J'} 2^{J'([\beta]+1)} \\ &\quad\leqslant C\,|h|^{\beta} \qquad (\text{because} \quad |J'-J| \leqslant C)\end{aligned}$$

It follows from the fact that $F_{J'}$ is bounded independently of J' that the second term is bounded by $C\Lambda_{J'}(x)$, so by $C2^{-J'(\beta-\epsilon)}$, hence by $C\,|h|^{\beta-\varepsilon}$.

The third term is bounded by

$$\sum_{j \geqslant J'} \sum_{i \in B_j(x)} |\lambda_i| \sum_{q=0}^{[\beta]} |I_i|^{-q} |h|^q \leqslant \sum_{j \geqslant J'} \Lambda_j(x) \sum_{q=0}^{[\beta]} 2^{qj} 2^{-qJ}$$
$$\leqslant C \sum_{j \geqslant J'} 2^{-\beta j} 2^{[\beta] j} 2^{-[\beta] J}$$
$$\leqslant C |h|^{\beta}$$

The proof of Theorem 4 is now achieved. ▮

5. COMPUTATION OF THE SPECTRUM OF SINGULARITIES

We only study case 2, for case 1 the restriction to $\mathscr{R}$ has no consequence on $d(\alpha)$ as in ref. 20. We consider a sequence $\{\mathscr{F}_n\}$ of finite partitions $\{V_i\}$ of $[0, 1[$ constituted of right-open intervals $\tilde{I}_{(i_1,\dots,i_n)}$ and $\tilde{H}_{(i_1,\dots,i_n)}$ deduced from the intervals $I_{(i_1,\dots,i_n)}$ and holes $H_{(i_1,\dots,i_n)}$, i.e.,

$$\mathscr{F}_0 = \{[0, 1[\},$$
$$\mathscr{F}_1 = \{[0, a_1[, [a_1, a_2[= \tilde{I}_1, [a_2, a_3[, [a_3, a_4[= \tilde{I}_2, [a_4, 1[\}$$
$$\mathscr{F}_2 = \{[0, a_1[, (\widetilde{l-H})_1, \tilde{I}_{11}, (\widetilde{m-H})_1, \tilde{I}_{12}, (\widetilde{r-H})_1, [a_2, a_3[, (\widetilde{l-H})_2,$$
$$\tilde{I}_{21}, (\widetilde{m-H})_2, \tilde{I}_{22}, (\widetilde{r-H})_2, [a_4, 1[\}$$

and so on. If $t \in [0, 1[$, let $\tilde{I}_n(t)$ be the element of $\mathscr{F}_n$ which contains t. Here, $\mathscr{F}_{n+1}$ is a refinement of $\mathscr{F}_n$ and $|\tilde{I}_n(t)| \to 0$ when $n \to +\infty$.

We denote dim the Hausdorff dimension limited by recovering of elements $\mathscr{F} = \bigcup_n \mathscr{F}_n$.

Theorem 5. If $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$, then $d(\alpha) = -\infty$.

Suppose that $\underline{K}(p, \varphi(p)) > 0$. Let $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ such that $\alpha < k$. If φ is derivable at p and if $\alpha = \varphi'(p)$ then

$$d(\alpha) = \alpha p - \varphi(p) = \inf_x (\alpha x - \varphi(x))$$

Proof of Theorem 5. The first point is straightforward because $E_F^{\alpha} = \varnothing$.

Let now $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ such that $\alpha < k$. If φ is derivable at p and if $\alpha = \varphi'(p) < k$ then the upper bound

$$d(\alpha) \leqslant \inf_x (\alpha x - \varphi(x))$$

follows from ref. 27, Theorem 1, pp. 780 (see also refs. 24 and 31).

Let now $\xi: \mathscr{F} \longrightarrow \mathbb{R}+$. Set

$$H(\xi)=\liminf_{\varepsilon\to 0}\left\{\sum_j \xi(V_j);\ \{V_j\} \text{ is a } \varepsilon\text{-recovering of } [0,1[\right\}$$

We say that ξ is a Frostman function if $H(\xi)>0$ and if 0 is an adherence value of the sequence $(\sup_{V\in\mathscr{F}_n}\xi(V))_n$.

Lemma 7 (cf. ref. 28). If ξ is a Frostman function, there exist a probability measure ν on $[0, 1[$, a constant $M>0$ and a number $\varepsilon>0$ such that

$$\forall V\in\mathscr{F},\quad |V|\leqslant\varepsilon: \qquad \nu(V)\leqslant M\xi(V) \tag{84}$$

We consider the following sets:

$$\tilde{E}^\alpha=\left\{t\in[0,1[;\ \lim_{n\to+\infty}\frac{\log|\lambda^1_{i_1}\cdots\lambda^n_{i_n}|}{\log|\tilde{I}_{(i_1,\dots,i_n)}(t)|}=\alpha\right\}$$

and

$$\bar{E}^\alpha=\left\{t\in[0,1[;\ \liminf_{n\to+\infty}\frac{\log|\lambda^1_{i_1}\cdots\lambda^n_{i_n}|}{\log|\tilde{I}_{(i_1,\dots,i_n)}(t)|}=\alpha\right\}$$

(with the ordinary precaution: $\frac{\log 0}{\log\eta}=+\infty$ if $\eta>0$).

For $n\in\mathbb{N}^*$ and $i=(i_1,\dots,i_n)\in\{1,2\}^n$ take

$$\xi(V)=\begin{cases}0 & \text{if}\quad V=\tilde{H}_{(i_1,\dots,i_n)}\\ |\lambda^1_{i_1}\cdots\lambda^n_{i_n}|^x\,|V|^{-\varphi(x)} & \text{if}\quad V=\tilde{I}_{(i_1,\dots,i_n)}\end{cases} \tag{85}$$

Since $\underline{K}(p,\varphi(p))>0$, then $H(\xi)>0$. In addition 0 is an adherence value of the sequence $(\sup_{V\in\mathscr{F}_n}\xi(V))_n$, because $\mathscr{F}_{n+1}$ is a refinement of $\mathscr{F}_n$ and $\forall V\in\mathscr{F}$, $|V|\longrightarrow 0$ when $n\to+\infty$. Thus ξ is a Frostman function, so the results of Lemma 7 hold.

Let

$$\hat{E}^\alpha=\left\{t\in[0,1[;\ \liminf_{n\to+\infty}\frac{\log\nu(\tilde{I}_n(t))}{\log|\tilde{I}_n(t)|}\geqslant\inf_x\,(\alpha x-\varphi(x))\right\}$$

It follows from (84) and (85) that $\tilde{E}^\alpha\subset\hat{E}^\alpha$. On the other hand, (84) yields that $\nu(\tilde{E}^\alpha)>0$. Whence, thanks to the Billingsly theorem,[32] we have

$$\dim\tilde{E}^\alpha\geqslant\inf_x\,(\alpha x-\varphi(x))$$

But

$$\tilde{E}^{\alpha} \subset \bar{E}^{\alpha} \subset E_F^{\alpha}$$

and

$$\bar{E}^{\alpha} = E_F^{\alpha} \backslash \mathcal{N} \qquad \text{with} \quad \mathcal{N} = \bigcap_{n=1}^{+\infty} \bigcup_{|i|=n} (I_i \backslash \tilde{I}_i)$$

Since $\mathcal{N}$ is countable then $d(\alpha) = \dim E_F^{\alpha} = \dim \bar{E}^{\alpha}$. Hence $d(\alpha) \geqslant \inf_x (\alpha x - \varphi(x))$. ∎

6. VALIDITY OF THE MULTI-FRACTAL FORMALISM

Theorem 6. If $\varphi(q)+1 < kq$ then $\eta(q) = \varphi(q)+1$.

Let $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ such that $\alpha < k$. If $\varphi(p)+1 < kp$, φ is derivable at p and $\underline{K}(p, \varphi(p)) > 0$, then for $\alpha = \varphi'(p)$, we have

$$d(\alpha) = \alpha p - \eta(p) + 1 = \inf_{q:\, \varphi(q)+1 < kq} (\alpha q - \eta(q) + 1)$$

Proof of Theorem 6. To compute $\eta(q)$, we have to estimate the size of the wavelet transform everywhere. For $i = (i_1, \ldots, i_n)$, consider

$$I_i(a) = I_i +]-a, a[$$

and

$$C_i = I_{(i_1, \ldots, i_{n-1})}(a) - I_{(i_1, \ldots, i_{n-1}, i_n)}(a)$$

If $i \in B_j$ and $a \leqslant |I_i|$ then $|I_i(a)| \sim |I_i|$, $|C_i| \leqslant |I_i|$ and inequalities (70) and (74) show that there exists $a \sim 2^{-j}$ and a point b of $I_i(a)$ for which the order of magnitude of $C_{a,b}(F)$ is $\Lambda_j(b)$.

We can also deduce the following lemma from (27) and an argument similar to the proof of Proposition 3.

Lemma 8. If $i \in B_j$, $a \sim |I_i|$ and $b \in I_i(a)$ then $|C_{a,b}(F)| \leqslant C\tilde{\Lambda}_j(b)$.
If $a \leqslant |I_i|$ and $b \in C_i$ then $|C_{a,b}(F)| \leqslant Ca^k\, |I_i|^{-k}\, \tilde{\Lambda}_j(b)$.

On the other hand, remark that

$$\frac{d}{db} C_{a,b} = \frac{1}{a} \tilde{C}_{a,b}$$

where $\tilde{C}_{a,b}$ is the wavelet transform due to ψ', and

$$\frac{d}{da} C_{a,b} = -\frac{1}{a} \check{C}_{a,b} + \frac{1}{a} C_{a,b}$$

where $\check{C}_{a,b}$ is the wavelet transform due to $x\psi'$.

We deduce that there exists an interval of length $\sim a$ on which the order of magnitude of $C_{a,b}(F)$ is $\Lambda_j(b)$. Thus if we denote by A_j the interval $[\frac{1}{2} 2^{-j}, 2^{-j}]$, then for each branch $i \in B_j$ there exists a ball of radius $\sim 2^{-j}$ in the time frequency half-space $\mathbb{R}^+ \times \mathbb{R}$, located near x_i and in frequency in the interval A_j and where

$$|C_{a,b}(F)| \geqslant C' |\lambda_i|$$

From Lemma 8 and the previous remark, we obtain

$$C' \sum_{i \in B_j} 2^{-2j} |\lambda_i|^q \leqslant \int_{A_j \times \mathbb{R}} |C_{a,b}(F)|^q \, da \, db \tag{86}$$

$$\leqslant C \sum_{i \in B_j} 2^{-2j} |\lambda_i|^q + O\left(2^{-j} \sum_{|I_i| \geqslant 2.2^{-j}} |I_i| \, |\lambda_i|^q \, 2^{-kqj} \, |I_i|^{-kq}\right)$$

$$\leqslant C 2^{-j} \left[\sum_{i \in B_j} 2^{-j} |\lambda_i|^q + O\left(\sum_{|I_i| \geqslant 2.2^{-j}} 2^{-kqj} |\lambda_i|^q \, |I_i|^{1-kq}\right)\right] \tag{87}$$

where $O(\cdot)$ is positive.

The properties of $C_n(x, y)$, implies that the function $\varphi: \mathbb{R} \longrightarrow \mathbb{R}$ given by (36) is non-decreasing and concave and satisfies $C(x, \varphi(x)) = 0$ for any x (see ref. 31). Then we have

$$0 = C(q, \varphi(q)) = \lim_{n \to \infty} \frac{1}{n} \log\left(\sum_{|i| = n} |\lambda_i|^q \, |I_i|^{-\varphi(q)}\right)$$

so we can write

$$\log\left(\sum_{|i| = n} |\lambda_i|^q \, |I_i|^{-\varphi(q)}\right) := o\left(\frac{1}{n}\right)$$

thus

$$\sum_{|i| = n} |\lambda_i|^q \, |I_i|^{-\varphi(q)} = e^{o(\frac{1}{n})}$$

Define

$$G(j) = \sum_{i \in B_j} |\lambda_i|^q \, |I_i|^{-\varphi(q)}$$

It follows from (18) that for $i \in B_j$,

$$j \frac{\log 2}{\log \theta^{-1}} - \frac{\log 2\mathscr{D}}{\log \theta^{-1}} < |i| \leqslant j \frac{\log 2}{\log \rho^{-1}} + \frac{\log \mathscr{D}}{\log \rho^{-1}} \tag{88}$$

Hence

$$C' e^{o(\frac{1}{j})} \leqslant G(j) \leqslant C j e^{o(\frac{1}{j})} \tag{89}$$

Therefore

$$\begin{aligned} 2^{-j} \sum_{i \in B_j} 2^{-j} |\lambda_i|^q &\sim 2^{-j} 2^{-(1+\varphi(q))\, j} \sum_{i \in B_j} |\lambda_i|^q \, |I_i|^{-\varphi(q)} \\ &= 2^{-j} 2^{-(1+\varphi(q))\, j} G(j) \\ &\leqslant C j e^{o(\frac{1}{j})} 2^{-j} 2^{-(1+\varphi(q))\, j} \end{aligned} \tag{90}$$

and

$$2^{-j} \sum_{i \in B_j} 2^{-j} |\lambda_i|^q \geqslant C' e^{o(\frac{1}{j})} 2^{-j} 2^{-(1+\varphi(q))\, j} \tag{91}$$

The term

$$2^{-j} O\left(\sum_{|I_i| \geqslant 2.2^{-j}} |\lambda_i|^q \, 2^{-kqj} \, |I_i|^{1-kq} \right) \tag{92}$$

is positive and bounded by

$$\begin{aligned} & C 2^{-j} 2^{-kqj} \sum_{|I_i| \geqslant 2.2^{-j}} |\lambda_i|^q \, |I_i|^{1-kq} \\ &\quad = C 2^{-j} 2^{-kqj} \sum_{l \leqslant j-2} \sum_{2^{-l-1} \leqslant |I_i| < 2^{-l}} |\lambda_i|^q \, |I_i|^{1-kq} \\ &\quad \sim C 2^{-j} 2^{-kqj} \sum_{l \leqslant j-2} 2^{-l(1-kq+\varphi(q))} \sum_{2^{-l-1} \leqslant |I_i| < 2^{-l}} |\lambda_i|^q \, |I_i|^{-\varphi(q)} \\ &\quad \leqslant C 2^{-j} 2^{-kqj} \sum_{l \leqslant j} 2^{-l(1-kq+\varphi(q))} G(l) \end{aligned}$$

thus if $\varphi(q) + 1 < kq$ then (92) is bounded by $C j e^{o(\frac{1}{j})} 2^{-(1+\varphi(q))\, j} 2^{-j}$.

Hence

$$C'2^{-j}e^{o(\frac{1}{j})}2^{-(1+\varphi(q))\,j} \leqslant \int_{A_j \times \mathbb{R}} |C_{a,b}|^q \, da \, db \leqslant Cj2^{-j}e^{o(\frac{1}{j})}2^{-(1+\varphi(q))\,j} \tag{93}$$

Using (93) and the fact that $\lim_{j\to\infty} \frac{o(\frac{1}{j})}{j} = 0$, we obtain

$$\limsup_{a\to 0} a^{-(1+\varphi(q))}e^{o(\frac{1}{\log a})} \int_{\mathbb{R}} |C_{a,b}|^q \, db \geqslant C' > 0 \tag{94}$$

and

$$\limsup_{a\to 0} \frac{a^{-(1+\varphi(q))}}{|\log a|} e^{o(\frac{1}{\log a})} \int_{\mathbb{R}} |C_{a,b}|^q \, db \leqslant C < +\infty \tag{95}$$

Therefore, if $\varphi(q)+1 < kq$ then $\eta(q) = \varphi(q)+1$. Whence, in view of Theorem 5, the multi-fractal formalism is valid. ▌

7. BOX DIMENSION OF THE GRAPH

We first recall a recent result of Jaffard (see ref. 33) in which he computed the box dimension of the graph of a function from its wavelet coefficients. Let S be a bounded subset in $\mathbb{R}^{m+1}$. Denote by $N(S, j)$ the number of dyadic cubes of side 2^{-j} necessary to cover S. The upper box dimension of S (also called fractal dimension) is

$$\overline{\dim_B}(S) = \limsup_{j\to\infty} \frac{\log N(S, j)}{j \log 2} \tag{96}$$

Let $2^{mj/2}\psi^{(i)}(2^j x - k)$, $(i = 1,..., 2^m - 1,\ j \in \mathbf{Z},\ k \in \mathbf{Z}^m)$ be an orthonormal basis of $L^2(\mathbb{R}^m)$. Let $f \in L^2(\mathbb{R}^m)$. We use a L^∞-normalization for wavelets, so that we write

$$f(x) = \sum_{i,j,k} C_{j,k}^{(i)} \psi^{(i)}(2^j x - k) \tag{97}$$

where

$$C_{j,k}^{(i)} = 2^{mj} \int f(t)\, \psi^{(i)}(2^j x - k)\, dt \tag{98}$$

Suppose furthermore that the $\psi^{(i)}$ and their gradients decay as x^{-2} at infinity. Denote by $\delta_{j,k}$ the dyadic cube $k2^{-j} + 2^{-j}[0, 1]^m$. In ref. 33, Jaffard proved the following result.

Proposition 4. Let $f: \mathbb{R}^m \longrightarrow \mathbb{R}$ be a function in $C^\varepsilon(\mathbb{R}^m)$ for $\varepsilon > 0$. Let Ω be a bounded domain of $\mathbb{R}^m$ with a Lipschitz boundary. If

$$w_j(\Omega) = \sum_{\delta_{j,k} \in \Omega} \sup_{\delta_{j',k'} \subset \delta_{j,k}} |C_{j,k}^{(i)}|$$

then

$$\overline{\dim_B}(\mathrm{Graph}_\Omega(f)) = \sup\left(m, 1 + \limsup_{j \to \infty} \frac{\log w_j(\Omega)}{j \log 2}\right)$$

But, our quasi-self-similar functions (we assume that the λ_i^n are reals) are not expressed in the form (97) because g is not necessary a wavelet. Therefore we can not directly apply Proposition 4. However, if $2^{-j} \leqslant a < 2.2^{-j}$ and $b = k2^{-j}$, then $\int_{\mathbb{R}^m} |C_{a,b}(F)| \, db \sim \frac{1}{2^{jm}} \sum_k |C_{j,k}(F)|$. This allows us to generalize Proposition 4 in the continuous form and then take advantage of the estimation of $C_{a,b}(F)$ established in Section 6.

Proposition 5. Let $f: \mathbb{R}^m \longrightarrow \mathbb{R}$ be a function in $C^\varepsilon(\mathbb{R}^m)$ for $\varepsilon > 0$. Let Ω is a bounded domain of $\mathbb{R}^m$ with Lipschitz boundary. Then

$$\overline{\dim_B}(\mathrm{Graph}_\Omega(f)) = \sup\left(m, 2 + \limsup_{a \to 0} \frac{\log \int w_{a,b} \, db}{\log \frac{1}{a}}\right)$$

where $w_{a,b}$ is the supremum of the $|C_{a',b'}|$ on a box centered on b and of side a.

We are now ready to compute the fractal dimension of the graph of a (real) quasi-self-similar function F.

Theorem 7. If $\varphi(1) + 1 < k$, then $\overline{\dim_B}(\mathrm{Graph}_I(F)) = \sup(1, 1 - \varphi(1))$.

Proof of Theorem 7. The function F is in $C^{\alpha_{\min} - \varepsilon}(\mathbb{R})$ for $\varepsilon > 0$. In addition, using the same technique as in Section 6, we deduce that if $\varphi(1) + 1 < k$ then $\int w_{a,b} \, db \sim a^{1+\varphi(1)}$.

So,

$$\limsup_{a \to 0} \frac{\log \int w_{a,b} \, db}{\log \frac{1}{a}} = -1 - \varphi(1)$$

Hence Theorem 7. ∎

8. SUMMARY, DETERMINISTIC AND RANDOM EXAMPLES, APPLICATIONS, AND PROSPECTS

We introduced the notion of quasi-self-similarity that extends the classical definition of self-similarity. Quasi-self-similar functions (26) are superposition of "similar" structures at different scales, reminiscent of some cascade models, but the weight associated to each structure can be chosen with some freedom as compared to the exact self-similar situation. In the spirit of wavelet analysis, the wavelet coefficients of quasi-self-similar functions are multiplicative weights that are allowed to depend on the generation. We used the special expression (26) of F and the iterated quasi-self-similar functional equation (27) to prove that the wavelet transform of F satisfies a similar "quasi-self-similar" functional equation, which enables us to estimate the size of the wavelet transform everywhere. This allows us to compute both the Hölder exponent $\alpha_F(x)$ for any point x and the Besov exponent $\eta(p)$. Using the "Frostman method," we established the validity of the multi-fractal formalism for quasi-self-similar functions: the singularity spectrum $d(\alpha)$ that is the Hausdorff dimension of the set of iso-Hölder regularity α can be obtained as the Legendre transform of $\eta(p)-1$. We also proved that $\eta(p)-1=\varphi(p)$ where the scaling exponent $\varphi(p)$ is defined from the asymptotic scaling of some multi-temperature partition functions.

The previous quasi-self-similar cascades are continuous, i.e., without preferable scale factors (continuous scale invariance). One can also distinguish discrete cascades that involve discrete scale invariance. Using arguments similar to those of this paper and those of the Jaffard's paper,[12] all the above results (with obvious modifications) remain valid for quasi-self-affine functions

$$F(x)=\sum_{n=0}^{\infty}\sum_{i=(i_1,\dots,i_n)\in\{1,\dots,d\}^n}\lambda_{i_1}^1\cdots\lambda_{i_n}^n\,g(S_i^{-1}(x)) \tag{99}$$

where the S_j's for $j=1,\dots,d$ are contractive similitudes satisfying conditions (13) and (14), the function g is C^k with all derivatives of order less than k having fast decay, and the F_N's corresponding to F satisfy (32).

Each time the multi-fractal formalism has been shown to hold, it was the consequence of some self-similarity (deterministic or statistic) either for the function or of its wavelet transform. It is therefore reasonable to conjecture that if a function satisfies some self-similarity condition, then the multi-fractal formalism is likely to hold. But it is impossible to state a reasonably general conjecture (one should be careful to avoid in such a statement the counterexamples exhibited by Ben Slimane in refs. 14–16).

If we allow the λ_i to be chosen along a whole tree, i.e., if F is written

$$F(x) = \sum_{n=0}^{\infty} \sum_{i=(i_1,\dots,i_n)\in\{1,\dots,d\}^n} \lambda_{(i_1,\dots,i_n)}\, g(S_i^{-1}(x)) \tag{100}$$

then in general F does not satisfy any quasi-self-similar equation. Therefore, it is not easy to determine analytically both the Hölder and Besov exponents. Besides, Jaffard[34] (resp. Arneodo *et al.*[35]) constructed some families of random (resp. deterministic) wavelet series of the form (100) for which the multi-fractal formalism does not hold.

Since quasi-self-similar (resp. quasi-self-affine) functions include a definitely larger class than the strict self-similar notion, one can expects a great number of applications of this work. The success of wavelet techniques in many fields of applications is due to the fact that, many signals, images, or mathematical functions f can be accurately represented in a wavelet basis. We can try to approximate f by a quasi-self-similar wavelet series. A quasi-self-similar wavelet series is built recursively on the dyadic grid of the orthogonal wavelet transform, involving only scales that range between a given large scale L and the scale 0 (excluded). Thus the corresponding fractal function $f(x)$ does not involve scales greater than L. Consider, for the sake of simplicity, a periodic function $f(x)$ of period L. Choose $L = 1$. The quasi-self-similar wavelet series is defined using a periodic orthonormal wavelet basis of $L^2_{\text{per}}([0, 1])$, i.e., the space of 1-periodic functions with finite energy.

Such a basis can be constructed using two functions $\phi(x)$ and $\psi(x)$ of $L^2_{\text{per}}([0, 1])$ (ψ is referred to as the analyzing wavelet) by means of translations and dilations of $\psi(x)$

$$\psi_{j,k} = 2^{j/2}\psi(2^j x - k), \qquad j \geqslant 0, \quad 0 \leqslant k \leqslant 2^j - 1 \tag{101}$$

One can prove that the so-obtained family of functions $\{\phi(x), \{\psi_{j,k}\}_{j,k}\}$ is an orthonormal basis of $L^2_{\text{per}}([0, 1])$ if $\phi(x)$ and ψ satisfy some conditions. Among these conditions, $\psi(x)$ should be localized around 0 and have N_ψ vanishing moments

$$\int_0^1 x^n \psi(x)\, dx = 0, \qquad \text{for all} \quad n < N_\psi \tag{102}$$

The wavelet coefficients $\{C_\phi, \{c_{j,k}\}_{j,k}\}$ of a function $f(x)$ are then defined (modulo a normalization factor) as the coefficients of f in the orthornomal wavelet basis

$$\begin{cases} c_\phi = \langle f, \phi \rangle = \int_0^1 f(x)\,\phi(x)\,dx, \\ c_{j,k} = 2^{j/2}\,\langle f, \psi_{j,k} \rangle = 2^{j/2} \int_0^1 f(x)\,\psi_{j,k}(x)\,dx = 2^j \int_0^1 f(x)\,\psi(2^j x - k)\,dx \end{cases} \tag{103}$$

Since $\{\phi(x), \{\psi_{j,k}\}_{j,k}\}$ is an orthornormal basis, one gets the reconstruction formula

$$f(x) = c_\phi \phi(x) + \sum_{j \geqslant 0} 2^{-j/2} \sum_{0 \leqslant k < 2^j} c_{j,k} \psi_{j,k}(x) \tag{104}$$

On the one hand, let us note that, since all the $\psi_{j,k}$ have at least one vanishing moment, c_ϕ essentially "captures" the mean value of f. This explains why it is often referred to as the approximation coefficient. On the other hand, assuming that the scale 1 "corresponds" to ψ, one can easily prove that $\psi_{j,k}(x)$ is localized around $x = x_{j,k}$ and corresponds to the scale a_j with

$$x_{j,k} = 2^{-j}k \qquad \text{and} \qquad a_j = 2^{-j} \tag{105}$$

Therefore, $c_{j,k}$ essentially "captures" the details of $f(x)$ around the point $x_{j,k}$ and at the scale a_j. They are referred to as the detail coefficients. These coefficients lie on a dyadic grid in the space-scale half-plane.

We built a quasi-self-similar wavelet series $\tilde{f}(x)$ by specifying its wavelet coefficients $\{c_{j,k}\}_{j,k}$ and c_ϕ. The $\{c_{j,k}\}_{j,k}$ are defined recursively in the following way:

$$\begin{cases} c_{0,0} = 1, \\ c_{j,2k} = W^{(l)}_{j-1,k} c_{j-1,k}, \\ c_{j,2k-1} = W^{(r)}_{j-1,k} c_{j-1,k} \end{cases} \tag{106}$$

for all j ($j \geqslant 1$) and k ($0 \leqslant k < 2^j$), where the $W^{(\varepsilon)}_{j,k}$ ($\varepsilon = l$, i.e., left or r, i.e., right) are reals between -1 and 1. We can try to approximate at best the original signal f by a quasi-self-similar wavelet series $\tilde{f}$ (associated to $g = \psi$ and the contractions $S_0(x) = x/2$ and $S_1(x) = x/2 + 1/2$) and thus deduce the multi-fractal properties of f.

8.1. Deterministic Examples

We now apply our results for some deterministic examples of self-similar functions and quasi-self-similar functions. We also give some examples of quasi-self-similar functions that are not self-similar and for which the function $\varphi(p)$ can be numerically estimated.

8.1.1. Some Self-Similar Functions

Let r be an integer such that $r \geqslant 2$, $S_i(x) = \frac{(x+i)}{r}$ for $i = 0,\dots, r-1$. Consider the self-similar function

$$F(x) = \sum_{i=0}^{r-1} \lambda_i F(S_i^{-1}(x)) + g(x)$$

where the function g is C^k with all derivatives of order less than k having fast decay. If $i = (i_1,\dots, i_n) \in \{0,\dots, r-1\}^n$, then $|I_i| = |S_i(I)| = \frac{1}{r^n}$. We have

$$C_n(x, y) = \frac{1}{n} \log\left(\sum_{|i|=n} |\lambda_i|^x\right) + \frac{1}{n} \log r^{ny} = \log(|\lambda_0|^x + \cdots + |\lambda_{r-1}|^x) + y \log r$$

Thus

$$\varphi(x) = -\frac{\log(|\lambda_0|^x + \cdots + |\lambda_{r-1}|^x)}{\log r} \tag{107}$$

Hence

$$\varphi(1) = -\frac{\log(|\lambda_0| + \cdots + |\lambda_{r-1}|)}{\log r}$$

Therefore, we recover the classical result of Falconer[36]: if $\varphi(1) + 1 < k$ then

$$\overline{\dim}_B(\mathrm{Graph}_I(F)) = \sup\left(1, 1 + \frac{\log(|\lambda_0| + \cdots + |\lambda_{r-1}|)}{\log r}\right)$$

We obtain similar results if r is not integer, $r \geqslant 2$ and if we only take S_0 and S_1. ▮

8.1.2. Some Quasi-Self-Similar Functions

Let r be an integer larger than 2. Let g be a C^k wavelet supported in I (resp. either a C^k compactly supported wavelet or a C^k function with all derivatives of order less than k having fast decay). The quasi-self-similar function associated to $S_i(x) = \frac{x+i}{r}$ for $i = 0,\dots, r-1$ is

$$F(x) = \sum_{j=0}^{+\infty} \lambda^1_{i_1(x)} \cdots \lambda^j_{i_j(x)} g(r^j x - r^{j-1} i_1(x) - \cdots - r i_{j-1}(x) - i_j(x))$$

$$\left(\text{resp. } F(x) = \sum_{j=0}^{+\infty} \sum_{(i_1,\dots, i_j) \in \{0,\dots, r-1\}^j} \lambda^1_{i_1} \cdots \lambda^j_{i_j} g(r^j x - r^{j-1} i_1 - \cdots - r i_{j-1} - i_j)\right)$$

These are wavelet series $\sum_{j=0}^{+\infty}\sum_l C_{j,l}\, g(r^j x - l)$ with $C_{0,0}=1$, $C_{j,l}=\lambda^1_{i_1}\cdots\lambda^j_{i_j}$ for $lr^{-j}=\frac{i_1}{r}+\cdots+\frac{i_j}{r^j}$ (resp. $C_{j,l}=\sum_{\substack{(i_1,\dots,i_j)\\ l=i_1r^{j-1}+\cdots+i_j}}\lambda^1_{i_1}\cdots\lambda^j_{i_j}$) and 0 elsewhere. We establish the following results:

If $r=2$ and $b(x):=\liminf_{j\mapsto\infty}\frac{\sup_{|l2^{-j}-x|<2.2^{-j}}\log|C_{j,l}|}{\log r^{-j}}<k$ then $\alpha_F(x)=b(x)$.

If $r\geqslant 3$ and $a(x)=\liminf_{n\to\infty}\frac{\log|\lambda^1_{i_1(x)}\cdots\lambda^n_{i_n(x)}|}{\log r^{-n}}<k$ then $\alpha_F(x)=a(x)$.

If $r\geqslant 2$ then

$$\varphi(p)=\liminf_{j\to\infty}\frac{\log(\sum_{(i_1,\dots,i_j)\in\{0,\dots,r-1\}^j}(|\lambda^1_{i_1}|\cdots|\lambda^j_{i_j}|)^p)}{\log r^{-j}} \tag{108}$$

$$=\liminf_{j\to\infty}\frac{\sum_{n=1}^j\log(\sum_{i=0}^{r-1}|\lambda^n_i|^p)}{\log r^{-j}} \tag{109}$$

If $\varphi(p)+1<kp$ then

$$\eta(p)=1+\liminf_{j\to\infty}\frac{\log(\sum_{(i_1,\dots,i_j)\in\{0,\dots,r-1\}^j}(|\lambda^1_{i_1}|\cdots|\lambda^j_{i_j}|)^p)}{\log r^{-j}} \tag{110}$$

$$=1+\liminf_{j\to\infty}\frac{\sum_{n=1}^j\log(\sum_{i=0}^{r-1}|\lambda^n_i|^p)}{\log r^{-j}} \tag{111}$$

If $1+\liminf_{j\to\infty}\frac{\sum_{n=1}^j\log(\sum_{i=0}^{r-1}|\lambda^n_i|)}{\log r^{-j}}<k$ then

$$\overline{\dim}_b(\mathrm{graph}_I(F))=\sup\left(1,\,1-\liminf_{j\to\infty}\frac{\sum_{n=1}^j\log(\sum_{i=0}^{r-1}|\lambda^n_i|)}{\log r^{-j}}\right)\quad\blacksquare$$

We now give some examples of functions that seem to be quasi-self-similar but are self-similar in reality. For $i=0,1$, let $\lambda^n_i=\lambda_i$ if n is even, β_i if n is odd. Assume that $|\lambda_i|\neq|\beta_i|$ for $i=0,1$. Consider the quasi-self-similar function associated to the sequences $(\lambda^n_0)_{n\geqslant1}$ and $(\lambda^n_1)_{n\geqslant1}$, to the contractions $S_0(x)=x/2$ and $S_1(x)=(x+1)/2$ and to a function g as above

$$F(x)=\sum_{j=0}^{+\infty}\sum_{(i_1,\dots,i_j)\in\{0,1\}^j}\lambda^1_{i_1}\cdots\lambda^j_{i_j}\,g(2^jx-2^{j-1}i_1-\cdots-2i_{j-1}-i_j)$$

Remark that $\lambda^1_{i_1}\cdots\lambda^j_{i_j}$ equals $(\beta_i)^{j/2}(\lambda_i)^{j/2}$ if j is even, and $(\beta_i)^{(j+1)/2}(\lambda_i)^{(j-1)/2}$ if j is odd. We can write

$$F(x)=\beta_0F_1(2x)+\beta_1F_1(2x-1)+g(x)$$

with

$$F_1(x)=\sum_{j=0}^{+\infty}\sum_{(i_1,\dots,i_j)\in\{0,1\}^j}\lambda_{i_1}^2\cdots\lambda_{i_j}^{1+j}\,g(2^jx-2^{j-1}i_1-\cdots-2i_{j-1}-i_j)$$

Remark that $\lambda_{i_1}^2\cdots\lambda_{i_j}^{1+j}$ equals $(\lambda_i)^{(j+1)/2}(\beta_i)^{(j-1)/2}$ if j is odd, and $(\lambda_i)^{j/2}(\beta_i)^{j/2}$ if j is even. Thus $F_1\neq F$ and so the function F is not self-similar under contractions S_0 and S_1. Nevertheless, since the function F_2 (written in (28) for $N=2$) is equal to F, then the quasi-self-similar equation (27) for $N=2$ for this function F becomes

$$\begin{aligned}F(x)&=\beta_0\lambda_0F(4x)+\beta_0\lambda_1F(4x-1)+\beta_1\lambda_0F(4x-2)\\&\quad+\beta_1\lambda_1F(4x-3)+\beta_0\,g(2x)+\beta_1\,g(2x-1)+g(x)\end{aligned}\tag{112}$$

As a consequence the function F is self-similar under the contractions $(x+i)/4$, $i=0,1,2,3$. Here we can compute $\varphi(p)$ using either formula (107) or (109). Formula (107) implies that

$$\varphi(p)=-\frac{\log(\beta_0^p+\beta_1^p)+\log(\lambda_0^p+\lambda_1^p)}{\log 4}\tag{113}$$

On the other hand

$$\sum_{n=1}^{j}\log(|\lambda_0^n|^p+|\lambda_1^n|^p)$$
$$=\begin{cases}\dfrac{j}{2}\log(|\lambda_0|^p+|\lambda_1|^p)+\dfrac{j}{2}\log(|\beta_0|^p+|\beta_1|^p) & \text{if } j \text{ is even}\\[2ex] \dfrac{(j-1)}{2}\log(|\lambda_0|^p+|\lambda_1|^p)+\dfrac{(j+1)}{2}\log(|\beta_0|^p+|\beta_1|^p) & \text{if } j \text{ is odd}\end{cases}\tag{114}$$

So formula (109) yields the same value for $\varphi(p)$. ∎

We now give some examples of quasi-self-similar functions that are not self-similar and for which the function $\varphi(p)$ can be numerically estimated. Take $\lambda_0^n=(-1)^n\,(\frac{1}{3}+\frac{1}{n+1})$ and $\lambda_1^n=(-1)^n\,(\frac{3}{4}+\frac{1}{n+1})$, then the associated quasi-self-similar function is not self-similar for the following reason: consider the function μ defined on $\mathscr{T}=\bigcup_n\mathscr{T}_n$ (where $\mathscr{T}_n=\{I_i;\ |i|=n\}$) by

$$\forall n\in\mathbb{N}^*\qquad\text{and}\qquad\forall I_i\in\mathscr{T}_n:\mu(I_i)=|\lambda_{i_1}^1\cdots\lambda_{i_n}^n|\tag{115}$$

For $I = I_{(i_1,\dots,i_n)} \in \mathcal{T}_n$ and $J = I_{(i_{n+1},\dots,i_{n+p})} \in \mathcal{T}_p$ we denote $IJ = I_{(i_1,\dots,i_{n+p})}$. Unlike the case (23) the function μ here is not quasi-Bernouilli; if μ were quasi-Bernouilli then

$$\exists C > 0; \quad \forall n, p \geqslant 1,\ \forall I \in \mathcal{T}_n,\ \forall J \in \mathcal{T}_p, \quad \frac{1}{C}\,\mu(I)\,\mu(J) \leqslant \mu(IJ) \leqslant C\mu(I)\,\mu(J)$$

This implies that

$$\forall n \geqslant 1, \quad \forall I \in \mathcal{T}_n, \quad \frac{1}{C} \leqslant \frac{\mu(II)}{(\mu(I))^2} \leqslant C$$

In particular

$$\forall n \geqslant 1, \qquad \frac{1}{C} \leqslant \left| \frac{\lambda_0^{n+1} \cdots \lambda_0^{2n}}{\lambda_0^1 \cdots \lambda_0^n} \right| \leqslant C \tag{116}$$

Set $U_n = |\frac{\lambda_0^{n+1} \cdots \lambda_0^{2n}}{\lambda_0^1 \cdots \lambda_0^n}|$, $Z_n = \log \frac{U_{n+1}}{U_n}$ and $S_{n-1} = \sum_{k=1}^{n-1} Z_k$. Clearly

$$U_n = U_1 \exp(S_{n-1}) \tag{117}$$

and

$$\begin{aligned} Z_n &= \log|\lambda_0^{2n+1}| + \log|\lambda_0^{2n+2}| - 2\log|\lambda_0^{n+1}| \\ &\simeq \frac{1}{(2n+1)} + \frac{1}{(2n+2)} - \frac{2}{(n+1)} \\ &\simeq -\frac{1}{n} \end{aligned}$$

Thus the series S_n (of general term Z_n) diverges to $-\infty$. Whence $\lim_{n\to\infty} U_n = 0$. So we have contradiction with the existence of $C > 0$ satisfying (116).

Now, formula (109) for $r = 2$ can be written as

$$\varphi(p) = \liminf_{j\to\infty} \frac{\mathcal{V}_1(p) + \cdots + \mathcal{V}_j(p)}{j} \tag{118}$$

where $\mathcal{V}_n(p) = -\frac{\log(|\lambda_0^n|^p + |\lambda_1^n|^p)}{\log 2}$. Note that if $\mathcal{V}_n(p)$ has a finite limit $\phi(p)$ when n goes to the ∞ then $\varphi(p) = \phi(p)$. For our example

$$\lim_{n\mapsto\infty} \mathcal{V}_n(p) = -\frac{\log((1/3)^p + (3/4)^p)}{\log 2}$$

So

$$\varphi(p) = -\frac{\log((1/3)^p + (3/4)^p)}{\log 2}$$

We now prove that $\underline{K}(p, \varphi(p)) > 0$ for any $p > 0$. For that we first prove the following lemma for

$$T_n := \sum_{|i|=n} |\lambda^1_{i_1} \cdots \lambda^n_{i_n}|^p \, |I_i|^{-\varphi(p)} \tag{119}$$

Lemma 9.

$$\forall p > 0, \qquad \liminf_{n \to \infty} T_n > 0 \tag{120}$$

Proof of the Lemma 9. We have (below p will be an exponent)

$$\begin{aligned}
T_n &= \sum_{|i|=n} |\lambda^1_{i_1} \cdots \lambda^n_{i_n}|^p \frac{1}{(\lambda_0^p + \lambda_1^p)^n} \quad \text{with } \lambda_0 = 1/3 \text{ and } \lambda_1 = 3/4 \\
&= \sum_{|i|=n} \frac{w^1_{i_1} \cdots w^n_{i_n}}{(\lambda_0^p + \lambda_1^p)^n} \qquad \text{with} \quad w^n_{i_n} = |\lambda^n_{i_n}|^p \\
&= \frac{\prod_{i=1}^n (w^i_0 + w^i_1)}{(\lambda_0^p + \lambda_1^p)^n} \\
&= \frac{\prod_{i=1}^n W_i}{(\lambda_0^p + \lambda_1^p)^n} \qquad \text{with} \quad W_i = w^i_0 + w^i_1 \\
&= \frac{\exp(\sum_{i=1}^n \log W_i)}{(\lambda_0^p + \lambda_1^p)^n} \\
&= \frac{\exp(\sum_{k=1}^n \log |\lambda^k_0|^p + |\lambda^k_1|^p)}{(\lambda_0^p + \lambda_1^p)^n} \\
&= \exp\left(\sum_{k=1}^n [\log(|\lambda^k_0|^p + |\lambda^k_1|^p) - \log \alpha]\right) \qquad \text{with} \quad \alpha = \lambda_0^p + \lambda_1^p \\
&= \exp\left(\sum_{k=1}^n U_k\right) \qquad \text{with} \quad U_n = \log\left[\frac{|\lambda^n_0|^p + |\lambda^n_1|^p}{\alpha}\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad = \log\left[\frac{(1/3 + \frac{1}{n+1})^p + (3/4 + \frac{1}{n+1})^p}{\alpha}\right]
\end{aligned}$$

We will study the convergence of the series of general term U_n. We have

$$\left(1/3+\frac{1}{n+1}\right)^p+\left(3/4+\frac{1}{n+1}\right)^p$$

$$=(1/3)^p+(3/4)^p+\frac{(1/3)^{p-1}\,p}{n+1}+\frac{(3/4)^{p-1}\,p}{n+1}+o\left(\frac{1}{n}\right)$$

$$=(1/3)^p+(3/4)^p+\frac{(1/3)^{p-1}\,p+(3/4)^{p-1}\,p}{n+1}+o\left(\frac{1}{n}\right)$$

and

$$U_n=\left(\frac{(1/3)^{p-1}\,p+(3/4)^{p-1}\,p}{(1/3)^p+(3/4)^p}\right)\frac{1}{n+1}+o\left(\frac{1}{n}\right) \tag{121}$$

Therefore $\forall p>0$, $\lim_{n\to\infty}\left(\sum_{k=1}^{n} U_k\right)=+\infty$. Whence $\liminf_{n\to\infty} T_n>0$. ∎

Proposition 6.

$$\forall p>0, \qquad \underline{K}(p,\varphi(p))>0 \tag{122}$$

Proof of Proposition 6. Let μ_n be the measure of Borel defined on $[0,1[$ by:

$$\mu_n(I_{(i_1,\dots,i_n)})=|\lambda_{i_1}^1\cdots\lambda_{i_n}^n|^p\left(\frac{1}{2^n}\right)^{-\varphi(p)} \tag{123}$$

Let ν_n be the probability measure given by $\nu_n=\frac{\mu_n}{T_n}$. Then ν_n converges weakly to a probability measure ν, i.e., $\forall f$ continuous on $[0,1]$, $\int f\,d\nu_n\longrightarrow\int f\,d\nu$.

Let $s\geqslant 0$ and $J=I_{(i_1,\dots,i_s)}$. Let $q\geqslant s$ and J_0^q and J_1^q the elements of $\mathcal{T}_n$ that are contiguous to J (without any loss of generality we can assume that they exist).

Let f be a continuous function with compact support such that $0\leqslant f\leqslant 1$, $f=1$ on J and Support$(f)\subset J_0^q\cup J\cup J_1^q$. Then

$$\nu(J)\leqslant\int f\,d\nu=\lim_{n\to\infty}\int f\,d\nu_n\leqslant\liminf_{n\to\infty}\left(\nu_n(J)+\nu_n(J_0^q)+\nu_n(J_1^q)\right) \tag{124}$$

Since $\mathscr{T}_{n+1}$ is a refinement of $\mathscr{T}_n$ then $I_{(i_1,\ldots,i_s)} = \bigcup_{i_{s+1},\ldots,i_n} I_{(i_1,\ldots,i_s,i_{s+1},\ldots,i_n)}$. So

$$\begin{aligned} \nu_n(I_{i_1,\ldots,i_s}) &= \frac{1}{T_n} \sum_{i_{s+1},\ldots,i_n} |\lambda^1_{i_1}|^x \cdots |\lambda^s_{i_s}|^x \, |\lambda^{s+1}_{i_{s+1}}|^x \cdots |\lambda^n_{i_n}|^x \left(\frac{1}{2^n}\right)^{-\varphi(x)} \\ &= \frac{|\lambda^1_{i_1}|^x \cdots |\lambda^s_{i_s}|^x}{T_s} \left(\frac{1}{2^s}\right)^{-\varphi(x)} \\ &= \frac{\mu_s(I_{(i_1,\ldots,i_s)})}{T_s} \end{aligned}$$

Since $\liminf_{n\to\infty} T_n > 0$ then there exists $\rho > 0$ such that for s large enough, $\nu_n(J) \leqslant \frac{1}{\rho} \mu_s(J)$. On the other hand for $n \geqslant q$, $\nu_n(J^q_0) = \nu_q(J^q_0)$ and $\nu_n(J^q_1) = \nu_q(J^q_1)$. It follows from (124) that for any q

$$\nu(J) \leqslant \frac{1}{\rho} \mu_s(J) + \nu_q(J^q_0) + \nu_q(J^q_1) \tag{125}$$

Without any loss of generality we can assume that $J^q_0 = I_{0,\ldots,0}$ and $J^q_1 = I_{1,\ldots,1}$. We will prove that $\lim_{q\to+\infty} \nu_q(J^q_0) = 0$. The other limit can be obtained by similar arguments. We have

$$\begin{aligned} \nu_q(J^q_0) &= \frac{(1/3+1/2)^p \cdots (1/3+\frac{1}{q+1})^p}{T_q} \left(\frac{1}{2^q}\right)^{-\varphi(p)} \\ &= \prod_{k=1}^{q} \frac{(1/3+\frac{1}{k+1})^p}{(1/3+\frac{1}{k+1})^p + (3/4+\frac{1}{k+1})^p} \end{aligned}$$

The general term of this product goes to $\frac{(1/3)^p}{(1/3)^p+(3/4)^p} < 1$. So

$$\lim_{q\to\infty} \nu_q(J^q_0) = 0$$

It follows from (125) that

$$\nu(J) \leqslant \frac{1}{\rho} \mu_s(J) \tag{126}$$

Let $\{I_i\}_i$ be a ε-recovering of $[0, 1[$. For $\varepsilon > 0$ small enough,

$$1 \leqslant \sum_i \nu(I_i) \leqslant \frac{1}{\rho} \sum_i \mu(I_i)^p \, |I_i|^{-\varphi(p)} \tag{127}$$

As a consequence $\underline{K}(p, \varphi(p)) > 0$. ∎

Therefore we get the following results:

- $\forall p > 0$, $\underline{K}(p, \varphi(p)) > 0$.
- $\alpha_{\min} = 2 - \frac{\log 3}{\log 2}$ and $\alpha_{\max} = \frac{\log 3}{\log 2}$.
- If $k \geqslant 2$, it is easy to see that $\forall p > 0$, $\varphi'(p) < k$. On the other hand $\forall p > 0$, $\varphi(p) + 1 < kp$. Thus for $\alpha = \varphi'(p_0) \in [2 - \frac{\log 3}{\log 2}, \frac{\log 3}{\log 2}]$, we have $d(\alpha) = \inf_{q>0}(\alpha q - \varphi(q)) = \varphi'(p_0)\, p_0 - \varphi(p_0)$. Moreover

$$\overline{\dim_B}(\mathrm{Graph}_I(F)) = 1 + \frac{\log(1/3 + 3/4)}{\log 2} \simeq 1{,}1154772$$

Clearly, similar results hold if we take $\lambda_0^n = (-1)^n\, (\mathscr{A} + \frac{1}{n})$ and $\lambda_1^n = (-1)^n\, (\mathscr{B} + \frac{1}{n})$ with $0 < \mathscr{A} < \mathscr{B} < 1$.

8.2. Random Quasi-Self-Similar Cascades

In ref. 37 (see also references therein), Arneodo *et al.* presented a first theoretical step towards a rigorous mathematical treatment of random cascading processes on the dyadic tree of their orthogonal wavelet coefficients. Arneodo *et al.* introduced a class of random quasi-self-affine functions using the orthogonal wavelet transform. To each random fractal function corresponds a random cascading process (which is a random quasi-self-similar wavelet series) on the dyadic tree of its orthogonal wavelet coefficients. Let us recall their work (for more details, see ref. 37). A random quasi-self-similar wavelet series (denoted a $\mathscr{W}$-cascade in ref. 37) is of the form (104) where the coefficient c_ϕ is chosen to be an arbitrary random variable and the $\{c_{j,k}\}_{j,k}$ are defined recursively as in (106) for all j ($j \geqslant 1$) and k ($0 \leqslant k < 2^j$) and where the $W_{j,k}^{(\varepsilon)}$ ($\varepsilon = l$ or r) are independent identically distributed (i.i.d.) real valued random variables. Since all the random variables $W_{j,k}^{(\varepsilon)}$ are i.i.d., we will omit the indexes j, k and (ε) and we will use W as the generic name for these variables. This recursive rule can be seen as a (quasi-self-similar) cascade process going from large scales (starting at scale 1) to smaller scales. It lies on a binary tree whose nodes are the wavelet coefficients and whose branches basically correspond (apart from the sign of the coefficients) to the same action of multiplying by W. Such a recursive rule is referred to a $\mathscr{W}$-cascade and $f(x)$ is referred to the function corresponding to the $\mathscr{W}$-cascade. Let us note that both a $\mathscr{W}$-cascade and its corresponding function are fully defined by the analyzing wavelet ψ, the laws of c_ϕ and W.

The so-obtained function $f(x)$ [assuming that the infinite sum in (104) converges] is quasi-self-similar in the sense that the law of a wavelet coefficient

$|c_{j_1,k}|$ at the scale 2^{-j_1} can be linked to the law of another wavelet coefficient $|c_{j_2,k'}|$ at the scale $2^{-j_2} > 2^{-j_1}$ using a multiplicative random variable depending only on the ratio of the two scales

$$|c_{j_1,k}| =_l |c_{j_2,k'}| X_{j_1-j_2}$$

where $=_l$ stands for the equality in law and where $X_n = |W_1 \cdots W_n|$ (the W_i's are i.i.d. real valued random variables with the same law as W). Thus, from a statistical point of view, the details of the function f at a scale a_1 are the same as the details at a scale a_2 up to a rescaling factor that depends only on a_1/a_2.

This was the "theoretical" description of a $\mathscr{W}$-cascade. Arneodo *et al.* then proved that the sum in (104) converges in some sense towards a (quasi-self-similar) random function $f(x)$. They proved that, for almost all realizations of the $\mathscr{W}$-cascade, (104) converges in $L^2_{\text{per}}([0,1])$, then they characterized some global regularity properties of the limit function. Let

$$\tau(q) = -\log_2 \mathscr{E}(|W|^q) - 1, \qquad \forall q \in \mathbb{R}$$

and $\mathscr{F}(\alpha)$ the Legendre transform of the function $\tau(q)$

$$\mathscr{F}(\alpha) = \inf_q \, (q\alpha - \tau(q))$$

Arneodo *et al.* proved that for almost all the realizations of the $\mathscr{W}$-cascades

$$\alpha_{\min} = \sup\{h < -\mathscr{E}(\log_2 |W|), \mathscr{F}(h) < 0\}$$

and

$$\alpha_{\max} = \inf\{h > -\mathscr{E}(\log_2 |W|), \mathscr{F}(h) < 0\}$$

Both spectra $d(\alpha)$ and $\mathscr{F}(\alpha)$ bring valuable information on the $\mathscr{W}$-cascade. The $d(\alpha)$ spectrum has been initially introduced for characterizing the singular behavior of deterministic fractal signals. We have seen that, for the large class of quasi-self-similar functions, the $d(\alpha)$ spectrum can be obtained using the wavelet based multifractal formalism. In the case of random $\mathscr{W}$-cascades, we actually get two spectra: the spectrum $d(\alpha)$ for each realization (which a priori depends on the realization) and the statistical spectrum $\mathscr{F}(\alpha)$ that characterizes the probability that a given singular behavior appears in a realization of the cascade. Thus, for instance, the maximum value of $\mathscr{F}(\alpha)$ corresponds to the most probable singular behavior in a realization of a $\mathscr{W}$-cascade. On the other hand, the negative values of $\mathscr{F}(\alpha)$ correspond to "rare" events that one should not expect to observe in almost all realizations. Arneodo *et al.* showed that, in the case of $\mathscr{W}$-cascades, the wavelet based multifractal formalism actually leads to a very reliable numerical estimation of the $\mathscr{F}(\alpha)$ spectrum. Arneodo *et al.* have shown mathematically and checked numerically on various computer synthetized signals, that very different statistical quantities such as the statistical spectrum, the self-similarity kernel and the correlation functions can be extracted directly from the fractal function using its wavelet decomposition (orthogonal or continuous) with an arbitrary analyzing wavelet. This mathematical study actually provides algorithms that are readily applicable to experimental situations. Recent applications of their methodology in the context of fully-developed turbulence have revealed the existence of a (nonscale invariant) log-normal cascading process underlying the turbulent velocity fluctuations. More surprising are the results of a similar investigation of financial times series. Underlying the fluctuations of the volatility (standard deviation) of the price variations, there exists a causal information cascade from large to small time scales that can be visualized with the wavelet representation. Let us emphasize that the fact that variations of prices over a one month scale influence in the future the daily price variations, is likely to be extraordinarily rich in consequences and this, not only for the fundamental understanding of the nature of financial markets, but also (and maybe more important) for practical applications. Indeed, the nature of the corrections across scales that are implied by this causal cascade has profound implications on the market risk, a problem of utmost concern for all financial institutions as well as individuals. These preliminary results are very promising as far as further experimental investigations of multiplicative cascade processes are concerned. Arneodo *et al.* thought that similar wavelet-based statistical analysis will lead to significant progress in fields other than hydrodynamic turbulence and finance.

ACKNOWLEDGMENTS

The authors are supported by the research project CMCU 99/F1506. The authors thank Fathi Ben Nasr for suggesting the problem studied in this paper and for stimulating discussions; and also Stéphane Jaffard for useful conversations. The authors thank the Department of Mathematics of Paris XII University, where part of this work was done, for its kind hospitality. The first author thanks Anouar Ben Mabrouk for discussions which he had with him at the beginning of the work. The authors also thank the referees for their comments, which greatly helped improve the presentation of the paper.

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